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The spectrum of skew-shift Schrödinger operators contains intervals

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Abstract

I prove that the spectrum of a skew-shift Schrödinger operator contains large intervals in the semi-classical regime. In the semi-classical limit, these intervals approach the range of the potential.

Keywords: Spectrum; Ergodic Schrödinger operators; Skew-shift

1. Introduction

In this paper, I consider the discrete Schrödinger operator $H_h = h\Delta + V$ on $\ell^2(\mathbb{Z})$ with potential

$$V(n) = f\left(\alpha n^2\right), \tag{1.1}$$

where α is a Diophantine number and f a one-periodic real-analytic function. In the semi-classical regime, that is for $h > 0$ sufficiently small, I show that the spectrum of the operator H_h contains large intervals (see Theorem 2.2), which approach the range of f as $h \to 0$. I will refer to the Schrödinger operator with potential given by (1.1) as the *skew-shift Schrödinger*

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operator, since its potential can be generated by evaluating a sampling function along the orbit of the skew-shift:

$$\begin{aligned} T_\alpha &: \mathbb{T}^2 \to \mathbb{T}^2, \\ T_\alpha(x, y) &= (x + 2\alpha, x + y) \pmod 1, \end{aligned} \tag{1.2}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit circle. It is expected that for all $h > 0$ the spectrum of the operator H_h is an interval, see the end of Chapter 15 in Bourgain's book [8] and the end of this introduction. Let me mention at this point that understanding the model above is of physical relevance due to its relation to the kicked-rotor problem, see Chapter 16 in [8].

My result extends the result of Bourgain from [5,6] that the spectrum has positive measure. As this result, the proof proceeds by analyzing a parametrization of the eigenvalues of finite restrictions, but I provide a mechanism that stops gaps from opening. One might wonder, why it is hard to prove that the spectrum consists of intervals, since it is true for simple examples like the free Laplacian Δ or periodic Schrödinger operators. Maybe, the most obvious obstructions are the results of Avila, Bochi, and Damanik [1,2], which show that generic potentials have Cantor spectrum. Before discussing results on Cantor spectrum further, I will comment on previous results showing interval spectrum.

Since the spectrum of random Schrödinger operators is the union of periodic spectra, these consist of intervals. See [17,20,21] for implementations of this. Also my results concerning the potential $V(n) = f(n^\rho)$ for $\rho > 0$ not an integer from [18] are an implementation of this fact, since they boil down to showing that there are arbitrarily long stretches of n, where $V(n) \approx f(x)$ for any $x \in \mathbb{T}$. It is also possible to construct limit-periodic examples with spectrum containing intervals see [14,22]. Most relevant to us is the work of Chulaevsky and Sinai [11], where they show that the spectrum of a two-frequency quasi-periodic Schrödinger operators is a single interval for a set of frequencies $\mathcal{A}$, which approaches all possible frequencies as $h \to 0$. However, Bourgain has shown in [6] that these models also exhibit gaps for arbitrary $h > 0$ but extreme frequencies.

My result that the spectrum of the skew-shift Schrödinger operator contains large intervals distinguishes these from one-frequency quasi-periodic Schrödinger operators, where the potential is given by

$$V_{\mathrm{QP}}(n) = f(\alpha n). \tag{1.3}$$

In fact for these Goldstein and Schlag [16] have shown that the spectrum is a Cantor set for almost every frequency α in the regime of $h > 0$ small. At this point, let me also point out that Avila and Jitomirskaya have solved the so called *Ten Martini Problem* [3,4], which asked to show that for $f(x) = 2\cos(2\pi x)$ the operator with potential given by (1.3) has Cantor spectrum for any irrational α and any $h > 0$.

As already mentioned, Avila, Bochi, and Damanik [1,2] have shown that for generic continuous sampling function and a large class of base transformations, one has Cantor spectrum. This result applies in particular to the skew-shift potential (1.1) with any irrational frequency α. I will further comment on the results from [1] when discussing optimality of my results.

If the sampling functions only takes finitely many values, it is known some generality, that the spectrum has zero Lebesgue measure, and thus is a Cantor set. This follows from the work of Damanik and Lenz [12,13].

Before coming to the technical discussion of the results, let me come to an aspect of the proof, I find surprising. If one iterates the skew-shift defined in (1.2), one finds

$$(T_\alpha)^n(x, y) = \big(x + 2n\alpha, y + nx + n(n-1)\alpha\big) \pmod 1. \tag{1.4}$$

One might expect that the relevant part of the dynamics is encoded in the αn^2 term, which is expected to behave like random variables [23]. However, we will not make use of this, but exploit the nx term, to obtain independence of events, which are far enough apart. This is possible, since x enters the problem as a *fast variable*, see Section 7 for the implementation of this fact.

However, the equidistribution properties of the sequence αn^2 enters the proof of the large deviation estimates, see [7,8,10].

Last, let me mention that it is easy to produce overwhelming numerical evidence for that the spectrum of $H_h = h\Delta + V$ is an interval for all $h > 0$. I will discuss this in Appendix A.

2. Statement of the results

I will now make the statement of my result precise. First, let me specify that I will assume the frequency α satisfies for some $c > 0$ the Diophantine condition

$$\|q\alpha\| \geqslant \frac{c}{q^2}, \tag{2.1}$$

for all integers $q \geqslant 1$, where $\|x\| = \operatorname{dist}(x, \mathbb{Z})$. Next, we will need the following result, which can be proven by the methods of Bourgain, Goldstein, and Schlag [10] or the ones of Bourgain [7].

Theorem 2.1. *(See [7,10].) There exists $h_0 = h_0(c, f) > 0$ such that for $0 < h < h_0$, we have that large deviation estimates for the Green's function hold.*

I will give a precise meaning to the large deviation estimates in Theorem 5.3. The requirement that the large deviation estimates hold is the first smallness condition on h, I impose.

The second smallness condition on $h > 0$ is required to verify the initial condition of the inductive scheme. Given $\delta > 0$ define a set of energies by

$$\mathcal{E}_\delta = \big\{E\colon \exists x\colon\ f(x) = E \text{ and } \big|f'(x)\big| \geqslant \delta\big\}. \tag{2.2}$$

This set is clearly a union of intervals. I am now ready to state the main result of the paper

Theorem 2.2. *There exists $h_1 = h_1(\delta, f) > 0$ such that for $0 < h < h_1$ we have*

$$\mathcal{E}_\delta \subseteq \sigma(H_h) \tag{2.3}$$

if the large deviation estimates for the Green's function hold.

Here, $\sigma(H_h)$ denotes the spectrum of the operator $H_h = h\Delta + V$, where V was given by (1.1). The smallness condition in this theorem does not depend on the Diophantine condition (2.1), but the condition that the large deviation estimates hold, imposes such a condition through Theorem 2.1. In Appendix B, I will demonstrate that

$$\begin{aligned}\min(f) - 2h \leqslant \min\bigl(\sigma(H_h)\bigr) \leqslant \min(f) - h,\\ \max(f) + h \leqslant \max\bigl(\sigma(H_h)\bigr) \leqslant \max(f) + 2h,\end{aligned} \tag{2.4}$$

which shows that Theorem 2.2 covers most of the spectrum.

I will prove a more precise result than Theorem 2.2. Recall that $f : \mathbb{T} \to \mathbb{R}$ is a non-constant real-analytic function. For $x, y \in \mathbb{T}$ and α Diophantine, we introduce the family of potentials

$$V_{x,y}(n) = f\bigl(\bigl(T_\alpha^n(x, y)\bigr)_2\bigr) = f\bigl(y + nx + n(n-1)\alpha\bigr). \tag{2.5}$$

Let now $h > 0$, then we introduce the family of skew-shift Schrödinger operators by

$$\begin{aligned}H_{h,x,y} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}),\\ H_{h,x,y}u(n) = h\bigl(u(n+1) + u(n-1)\bigr) + V_{x,y}(n)u(n).\end{aligned} \tag{2.6}$$

In short notation, we have $H_{h,x,y} = h\Delta + V_{x,y}$. We also note that $H_h = H_{h,\alpha,0}$. The more precise result is

Theorem 2.3. *There exists $h_1 = h_1(\delta, f) > 0$. Let $0 < h < h_1$ such that the large deviation estimates for the Green's function hold. Then for $E \in \mathcal{E}_\delta$, there exist $(x, y) \in \mathbb{T}^2$ such that E is an eigenvalue of $H_{h,x,y}$.*

By minimality of the skew-shift, we have that $\sigma(H_{h,x,y}) = \sigma(H_{h,\tilde{x},\tilde{y}})$ for any $x, y, \tilde{x}, \tilde{y} \in \mathbb{T}^2$. Using this, it is easy to show that Theorem 2.3 implies Theorem 2.2. Minimality of the skew-shift even implies that the set of $(x, y) \in \mathbb{T}^2$ such that E is an eigenvalue of $H_{h,x,y}$ is dense. However, it follows from the general theory of ergodic Schrödinger operators, that this set has zero measure, see Theorem 5.3 in [24].

It should be mentioned here that it is an open question whether the operator $H_{h,x,y}$ has pure point spectrum for almost every (x, y) or not. The results of Bourgain, Goldstein, and Schlag only imply pure point spectrum for almost every frequency α. So in some sense, Theorem 2.3 exhibiting at least one eigenvalue for some x, y is a step towards proving that $H_{h,x,y}$ has pure point spectrum for almost every $(x, y) \in \mathbb{T}^2$.

2.1. Optimality of the results

It might seem that the choice of potential in (2.5) is somewhat arbitrary, since we assume that the function only depends on the second coordinate. However, when one considers potentials of the more general form

$$V_{x,y}(n) = g\bigl(T^n(x, y)\bigr) \tag{2.7}$$

for a real-analytic function $g : \mathbb{T}^2 \to \mathbb{R}$, one faces obstruction to the spectrum being an interval.

Consider g of the form

$$g(x, y) = 2\cos(2\pi x) + \kappa \cos(2\pi y) \tag{2.8}$$

for some small $\kappa > 0$. For $\kappa = 0$, the operator H_h will just be the Almost–Mathieu operator, which is known to have Cantor spectrum, in particular it has at least one gap of size at least η for $\eta > 0$ sufficiently small. Hence, if $\kappa < \frac{\eta}{2}$ also the operator with skew-shift potential depending non-trivially on the second coordinate has at least one gap in its spectrum.

For fixed $h > 0$ and any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$, Avila, Bochi, and Damanik have shown in [1] that there exists a continuous function $f_1 : \mathbb{T}^2 \to \mathbb{R}$ such that $\|f - f_1\|_{L^\infty(\mathbb{T}^2)}$ is arbitrarily small and the Schrödinger operator $H_{h,1} = h\Delta + V_1$ with

$$V_1(n) = f_1\bigl(T^n(0,0)\bigr) \tag{2.9}$$

has Cantor spectrum. In particular, the spectrum contains a gap of size 2η, that is there is some E_0 such that

$$\sigma(h\Delta + V_1) \cap [E_0 - \eta, E_0 + \eta] = \{E_0 - \eta, E_0 + \eta\}. \tag{2.10}$$

It is classical that there exists now an analytic function f_2 such that $\|f_1 - f_2\|_{L^\infty(\mathbb{T}^2)} \leqslant \frac{\eta}{2}$. Then standard perturbation theory shows that

$$\sigma(h\Delta + V_2) \cap \left(E_0 - \frac{\eta}{2}, E_0 + \frac{\eta}{2}\right) \neq \emptyset, \tag{2.11}$$

where $V_2(n) = f_2(T^n(0,0))$. Hence, this operator has a gap in the spectrum.

However, an inspection of the argument of my proof, shows that my result is stable under perturbing the sampling function f in the $C^1(\mathbb{T}^2)$ topology as long as the large deviation estimates continue to hold. For this it is necessary, that the domain of analyticity of f stays the same.

2.2. *Discussion of the proof*

I will now try to explain the main ideas in this paper. Let me begin by pointing out that checking the initial condition of the multi-scale scheme in Section 4, is done by a computation similar to the one I used in [19], to show that all gaps $[E_-, E_+]$ of $\sigma(H_h)$ must satisfy $E_+ - E_- = O(h^2)$ as $h \to 0$.

As mentioned above the proof proceeds by a multi-scale scheme. This scheme bears some similarities to the one used by Bourgain in [5,6] to prove that the measure of the spectrum of quasi-periodic Schrödinger operators in the localization regime is positive. A key difference is that the arguments of this paper use analytic perturbation theory to show that a fixed number E_0 belongs to the spectrum, see Section 9. This is necessary, since extending an eigenvalue from scale to scale slightly perturbs it.

Maybe the key insight was that since the nth iterate of the skew-shift is

$$T_\alpha^n(x, y) = \bigl(x + 2n\alpha, y + nx + n(n-1)\alpha\bigr) \pmod 1, \tag{2.12}$$

one has that x enters the problem as a *fast variable*, since it gets multiplied by n in the second coordinate. This realization will allow us to prove an elimination of x argument in Section 7, which is used to eliminate *double resonances*. At this point let me also mention that the argument of Section 7 is an adaptation of the frequency elimination argument of Bourgain and Goldstein from [9].

As a further point of interest, let me point out that of the arguments to prove the presence of gaps in the spectrum, my argument is most related to the one of Goldstein and Schlag [16], see also [15] for a non-technical discussion. My argument shows that there are always simple resonances and that I can eliminate double resonances, whereas Goldstein and Schlag show that there are certain simple resonances, that must also be double resonances, and not triple. The formation of double resonances then implies that gaps must open.

I furthermore wish to point out, what the methods of this paper would yield for the one-frequency quasi-periodic model, so consider the potential

$$V_{x,\alpha}^{\mathrm{QP}}(n) = f(n\alpha + x) \tag{2.13}$$

and the associated Schrödinger operator $H_{h,x,\alpha}^{\mathrm{QP}} = h\Delta + V_{x,\alpha}^{\mathrm{QP}}$. Now, as in [9], α will play the role of a fast-variable. Translating the statement of Theorem 2.3, one obtains that for every $E \in \mathcal{E}_\delta$, there exists a set $\mathcal{A} = \mathcal{A}(E, h)$ such that:

(i) $|\mathcal{A}| \to 1$ as $h \to 0$.
(ii) For $\alpha \in \mathcal{A}$, there exists $x \in \mathbb{T}$ such that E is an eigenvalue of $H_{h,x,\alpha}^{\mathrm{QP}}$.

It is clear that this is compatible with quasi-periodic Schrödinger operators having gaps in their spectrum.

2.3. *A non-technical description of the proof*

Having now explained the main ideas behind the proof, let me explain some of the details. Denote by $H_{h,x,y}^{[-N,N]}$ the restriction of $H_{h,x,y}$ to $\ell^2(\{-N, \dots, N\})$. Given $E_0 \in \mathcal{E}_\delta$, we will construct inductively a sequence N_j and curves $\xi_j : \mathbb{T} \to \mathbb{T}$ such that for a positive measure set $\mathfrak{X}_j$, we have

$$E_0 \text{ is an eigenvalue of } H_{h,x,\xi_j(x)}^{[-N_j,N_j]} \tag{2.14}$$

whenever $x \in \mathfrak{X}_j$. The main problem with this approach is to pass from scale to scale. In order to discuss some aspects of this problem, I have included Fig. 1, which shows the set of (x, y) such that

$$0 \text{ is an eigenvalue of } H_{0.1,x,y}^{[-1,1]} \tag{2.15}$$

for the potential

$$V_{x,y}(n) = \cos\bigl(2\pi\bigl(\sqrt{2}n(n-1) + nx + y\bigr)\bigr). \tag{2.16}$$

One should notice in this figure that there are parts of two almost straight segments around $y = 0.25$ and $y = 0.75$. These correspond to the fact that

$$V_{x,y}(0) = 0, \quad y \in \{0.25, 0.75\}. \tag{2.17}$$

The interruptions in these lines can be identified with the set of x such that there exists

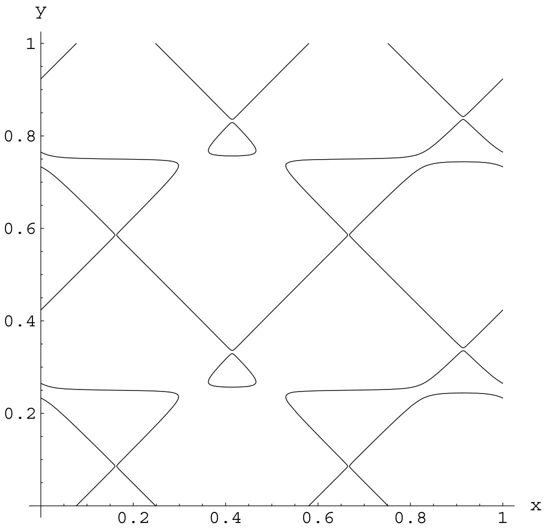

Fig. 1. The (x, y) such that $0 \in \sigma(H^{[-1,1]}_{0.1,x,y})$.

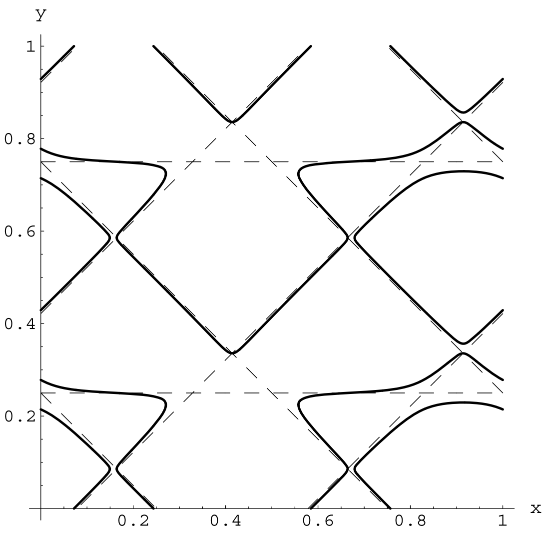

Fig. 2. The (x, y) such that $0 \in \sigma(H^{[-1,1]}_{0.2,x,y})$ in thick. The (x, y) such that $V_{x,y}(n) = 0$ for $n = -1, 0, 1$ are dashed.

$n \in \{-1, 0, 1\}$ such that

$$V_{x,0.25}(n) = 0 \quad \text{respectively } V_{x,0.75}(n) = 0. \tag{2.18}$$

This is illustrated in Fig. 2, where these lines are shown dashed. Making these assertions precise is the content of the first step in the proof of the initial condition given in Section 4.

I have also included Fig. 3, which shows the same situation as Fig. 1 except for $N = 2$ instead of $N = 1$.

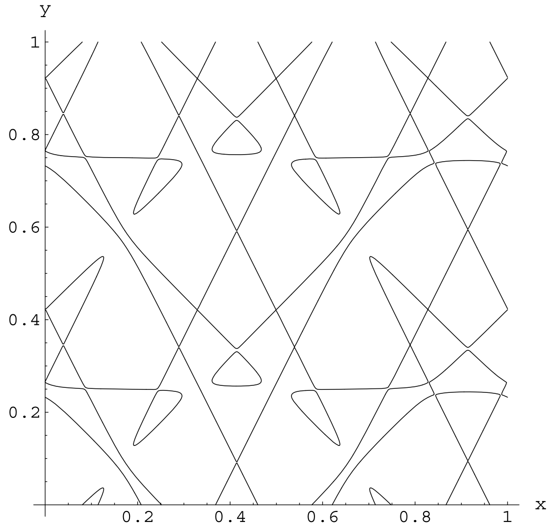

Fig. 3. The (x, y) such that $0 \in \sigma(H_{0.1,x,y}^{[-2,2]})$.

The explanations so far explain, why there are eigenvalues close to the line $y = 0.25$ for many x. Let me now mention that using analytic perturbation theory, one can construct a function ξ_1 such that for many x, we have

$$0 \text{ is in the spectrum of } H_{h,x,\xi_1(x)}^{[-N_1,N_1]} \tag{2.19}$$

for some $N_1 \geqslant 1$. This function ξ_1 will satisfy that $|\xi_1'(x)|$ and $|\xi_1(x) - 0.25|$ are both small.

In order to pass to the scale N_2, we will exploit that x is a fast variable and the large deviation estimates for the Green's functions. These will allow us to show that for

$$-N_2 \leqslant n \leqslant -\frac{N_1}{10}, \qquad \frac{N_1}{10} \leqslant n \leqslant N_2, \tag{2.20}$$

we have

$$\operatorname{dist}\big(0, \sigma\big(H_{h,x,\xi(x)}^{n+[-N_1,N_1]}\big)\big) \text{ is not too small.} \tag{2.21}$$

This is what is called *elimination of double resonances*. Using this, we are able to show that for many x, we have

$$0 \text{ is extremely close to the spectrum of } H_{h,x,\xi_1(x)}^{[-N_2,N_2]}. \tag{2.22}$$

Then, we construct ξ_2 similarly to ξ_1 and repeat the process.

In order to show that E_0 is indeed an eigenvalue of $H_{h,x,y}$ for some y, we will show that there are many x that are good for all j. Furthermore, for these we have that

$$\xi_j(x) \to \xi_\infty(x) \tag{2.23}$$

and also the eigenfunctions ψ_j corresponding to the eigenvalue E_0 are convergent to some ψ_∞. This implies that

$$H_{h,x,\xi_\infty(x)}\psi_\infty = E_0\psi_\infty \tag{2.24}$$

and thus that E_0 is an eigenvalue of $H_{h,x,y}$ for some y.

3. Outline of the proof

In this section, we explain the inductive construction, which we use in the proof. The following two sections contain the explanations of how to do check the initial condition and how to obtain the induction step. The proof is concerned with parameterizing isolated eigenvalues

Definition 3.1. Let $\varepsilon > 0$, A a self-adjoint operator, and $E \in \mathbb{R}$. E is an ε-isolated eigenvalue of A, if

$$\sigma(A) \cap [E-\varepsilon, E+\varepsilon] = \{E\} \tag{3.1}$$

and E is simple.

An eigenvalue E of A is simple if $\ker(A-E)$ is one-dimensional. In the setting of one-dimensional Schrödinger operators this condition is always satisfied, see Remark 1.10 in [24].

Isolated eigenvalues are important, since they behave well under perturbations. This can for example be seen in Lemma 9.2. The conclusions of this lemma can be summarized that an ε-isolated eigenvalue is stable under perturbations of A of size ε^2. Next, we define what we mean by a parametrization:

Definition 3.2. Let $\xi : \mathbb{T} \to \mathbb{T}$, $\mathfrak{X} \subseteq \mathbb{T}$ be a continuously differentiable function, $\varepsilon > 0$, $L \in (0, \frac{1}{3})$, and $M \geqslant 0$. We say that $(\xi, \mathfrak{X})$ is an (ε, L)-parametrization of the eigenvalue E_0 of $H_{h,\bullet}^{[-M,M]}$, if:

(i) For $x \in \mathfrak{X}$, we have

$$E_0 \text{ is an } \varepsilon\text{-isolated eigenvalue of } H_{h,x,\xi(x)}^{[-M,M]}. \tag{3.2}$$

(ii) We have $|\mathfrak{X}| \geqslant \frac{1}{\sqrt{\max(M,1)}}$.
(iii) We have $\|\xi'\|_{L^\infty(\mathbb{T})} \leqslant L$.

In order to examine this definition, let us look at the simplest example of $M = 0$. Then, we have that $H_{h,x,y}^{[-0,0]}$ is just the multiplication operator by $f(y)$. Hence, if we define

$$E_0 = f(y_0), \qquad \xi_0(x) = y_0, \qquad \mathfrak{X}_0 = \mathbb{T}, \qquad M_0 = 0, \qquad \varepsilon_0 = 1 \tag{3.3}$$

we have that

$$(\xi_0, \mathfrak{X}_0) \text{ is an } (\varepsilon_0, 0)\text{-parametrization of the eigenvalue } E_0 \text{ of } H_{h,\bullet}^{[-M_0,M_0]}. \tag{3.4}$$

The essential part of the induction scheme will be to show that given a parametrization at scale M, we can extend it to a parametrization at scale $R \approx e^{M^c}$ for some positive $c > 0$. However, this alone will not carry a sufficient amount of informations, we will also want that the eigenfunctions will have something to do with each other. For this, we introduce

Definition 3.3. Let $(\xi_j, \mathfrak{X}_j)$ be (ε_j, L_j)-parameterizations of the eigenvalue E_0 of $H_{h,\bullet}^{[-M_j,M_j]}$ for $j = 1, 2$.

$(\xi_2, \mathfrak{X}_2)$ is said to be a δ-extension of $(\xi_1, \mathfrak{X}_1)$, if:

(i) $\mathfrak{X}_2 \subseteq \mathfrak{X}_1$.
(ii) $\varepsilon_2 < \varepsilon_1$, $M_2 \geqslant M_1$.
(iii) $L_2 \leqslant L_1 + \delta$ and $\|\xi_1 - \xi_2\|_{L^\infty(\mathfrak{X}_2)} \leqslant \delta$.
(iv) Let $x \in \mathfrak{X}_2$ and $\varphi_j \in \ell^2([-M_j, M_j])$ normalized eigenfunctions of $H_{h,x,\xi_j(x)}^{[-M_j,M_j]}$ corresponding to the eigenvalue E_0. We have for some $|a| = 1$

$$\|\varphi_1 - a\varphi_2\|_W \leqslant \delta. \tag{3.5}$$

Here, we take $W(n) = 1 + n^2$ and we define the norm

$$\|u\|_W = \left(\sum_{n\in\mathbb{Z}} W(n)\big|u(n)\big|^2\right)^{\frac{1}{2}} \tag{3.6}$$

which is always well defined, since for us u and v are non-zero for only finitely many n. The reason for adding the weight W is that, we will want to control $\langle\psi, \partial_x V\psi\rangle$, where the norm of $\partial_x V$ as an operator on $\ell^2([-N, N])$ grows like N.

The operators $\partial_x V$ and $\partial_y V$ are defined by

$$(\partial_x V_{x,y}u)(n) = nf'\big(y + nx + n(n-1)\alpha\big)u(n), \tag{3.7}$$

$$(\partial_y V_{x,y}u)(n) = f'\big(y + nx + n(n-1)\alpha\big)u(n), \tag{3.8}$$

where the (x, y) are always the ones so that the ψ in $\langle\psi, \partial_x V\psi\rangle$ is an eigenfunction of $H_{h,x,y}^{[-M,M]}$.

We define in the following

$$d = \frac{1}{10}\max\big(\big|f'(y_0)\big|, 1\big). \tag{3.9}$$

In order to understand one way, in which we will use Definition 3.3, we prove

Lemma 3.4. *Let $(\xi, \mathfrak{X})$ be a δ-extension of $(\xi_0, \mathfrak{X}_0)$. Let $x \in \mathfrak{X}$ and ψ be a normalized eigenfunction of $H_{h,x,\xi(x)}^{[-M,M]}$ corresponding to the eigenvalue $E_0 = f(y_0)$. Assume that $\delta \leqslant \frac{d^5}{2C_1}$ where $C_1 = 10\|f'\|_{L^\infty(\mathbb{T})}$. Then we have that*

$$\big|\langle\psi, V_x\psi\rangle\big| \leqslant \frac{1}{4}d^5, \qquad \big|\langle\psi, V_y\psi\rangle\big| \geqslant 4d. \tag{3.10}$$

Proof. Let $x \in \mathfrak{X}$. By condition (iv) of Definition 3.3, there exists a choice of a normalized eigenfunction ψ of $H_{h,x,\xi(x)}^{[-M,M]}$ corresponding to the eigenvalue E_0 such that

$$\psi(0) \geqslant 1-\delta, \qquad \sum_{n\in\mathbb{Z}} |n| \left|\psi(n)\right|^2 \leqslant \delta.$$

This implies that

$$\left|\langle\psi, V_x\psi\rangle\right| \leqslant C_1\delta, \qquad \left|\langle\psi, V_y\psi\rangle\right| \geqslant 10d - (C_1+10)\delta.$$

The claim now follows by the choice of δ. $\square$

This lemma shows, how we will use the condition the parametrization extends $(\xi_0, \mathfrak{X}_0)$. We will use these conditions to obtain some control on the ξ in the parametrization, in particular that $\|\xi'\|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{3}$, which we need to eliminate double resonances.

We will pass from scale to scale using the next theorem.

Theorem 3.5. *Let M be large enough and assume that the large deviation estimate holds. Furthermore, assume for $\varepsilon = \mathrm{e}^{-M^{\frac{1}{50}}}$ that*

$$(\xi, \mathfrak{X}) \textit{ is an } (\varepsilon, L)\textit{-parametrization of the eigenvalue } E_0 \textit{ of } H_{h,\bullet}^{[-M,M]} \tag{3.11}$$

that $\frac{d^5}{2C_1}$-extends $(\xi_0, \mathfrak{X}_0)$ and $L + \varepsilon \leqslant \frac{1}{3}$. Define $R = \lfloor \mathrm{e}^{M^{\frac{1}{1000}}} \rfloor$. Then there exists $(\hat{\xi}, \widehat{\mathfrak{X}})$ such that

$$(\hat{\xi}, \widehat{\mathfrak{X}}) \textit{ is a } \left(\frac{1}{1000}\varepsilon, \hat{L}\right)\textit{-parametrization of the eigenvalue } E_0 \textit{ of } H_{h,\bullet}^{[-R,R]} \tag{3.12}$$

with $\hat{L} = L + \varepsilon$ and for $\eta = \mathrm{e}^{-\frac{1}{100}M}$

$$(\hat{\xi}, \widehat{\mathfrak{X}}) \textit{ is an } \eta\textit{-extension of } (\xi, \mathfrak{X}) \textit{ to scale } R. \tag{3.13}$$

Since M needs to be large, $(\xi_0, \mathfrak{X}_0)$ does not satisfy the assumptions of this theorem. So, we will need

Theorem 3.6. *Let $M \geqslant 1$. Then there exists $h_2 = h_2(M, f, \delta) > 0$ such that for $0 < h < h_2$ and $E_0 \in \mathcal{E}_\delta$, there exists an $h^{\frac{1}{500}}$-parameterization $(\xi_1, \mathfrak{X}_1)$ at scale M that $h^{\frac{1}{10}}$-extends $(\xi_0, \mathfrak{X}_0)$.*

We now begin the proof of Theorem 2.3. Choose M so large that Theorem 3.5 holds, finitely many additional largeness conditions might be imposed below. For $M_1 = M$, we define a sequence

$$M_{j+1} = \left\lfloor \mathrm{e}^{(M_j)^{\frac{1}{1000}}} \right\rfloor \tag{3.14}$$

and

$$\varepsilon_j = \mathrm{e}^{-(M_j)^{\frac{1}{50}}}, \qquad L_{j+1} = L_j + \varepsilon_j. \tag{3.15}$$

For M large enough, $\sum_{\ell=j+1}^{\infty} \varepsilon_\ell \leqslant \varepsilon_j$, $L_j \leqslant \frac{1}{3}$, and $\varepsilon_1 \leqslant \frac{1}{6}$

We will now inductively construct $(\xi_j, \mathfrak{X}_j)$ such that

$$(\xi_j, \mathfrak{X}_j) \text{ is an } (\varepsilon_j, L_j)\text{-parametrization of the eigenvalue } E_0 \text{ of } H_{h,\bullet}^{[-M_j, M_j]} \tag{3.16}$$

and

$$(\xi_{j+1}, \mathfrak{X}_{j+1}) \text{ is an } \varepsilon_{j+1}\text{-extension of } (\xi_j, \mathfrak{X}_j). \tag{3.17}$$

We construct $(\xi_1, \mathfrak{X}_1)$ using Theorem 3.6. We can require here that h is small enough such that

$$h^{\frac{1}{500}} \leqslant \varepsilon_1, \qquad h^{\frac{3}{2}} \leqslant \frac{d^5}{10C_1}. \tag{3.18}$$

We now see that the assumptions of Theorem 3.5 hold for $(\xi_1, \mathfrak{X}_1)$, so we can construct $(\xi_2, \mathfrak{X}_2)$. Using the following lemma, one can now construct $(\xi_{j+1}, \mathfrak{X}_{j+1})$ from $(\xi_j, \mathfrak{X}_j)$ using Theorem 3.5.

Lemma 3.7. *Let $\ell < j$, then*

$$(\xi_\ell, \mathfrak{X}_\ell) \textit{ is a } 2\varepsilon_\ell\textit{-extension of } (\xi_j, \mathfrak{X}_j). \tag{3.19}$$

Proof. This follows by being an extension is transitive. □

We will obtain this way a sequence of compact subsets

$$\mathfrak{X}_1 \supseteq \mathfrak{X}_2 \supseteq \mathfrak{X}_3 \supseteq \mathfrak{X}_4 \supseteq \cdots \tag{3.20}$$

of $\mathbb{T}$. Since the $\mathfrak{X}_j$ are compact, we have for some x_∞ that

$$x_\infty \in \bigcap_{j=1}^{\infty} \mathfrak{X}_j. \tag{3.21}$$

Define $y_j = \xi_j(x_\infty)$. We clearly also have $y_j \to y_\infty$ in $\ell^2(\mathbb{Z})$. By condition (iv) of Definition 3.3, we can choose eigenfunctions ψ_j of $H_{h,x,y_j}^{[-M_j, M_j]}$ corresponding to the eigenvalue E_0, which form a Cauchy sequence. By continuity, we have

$$H_{h,x_\infty,y_\infty}\psi_\infty = E_0\psi_\infty, \tag{3.22}$$

where $\psi_\infty = \lim_{j\to\infty} \psi_j$. This finishes the proof of Theorem 2.3.

4. Proof of the initial condition

In this section, we prove the initial condition, that is Theorem 3.6. In order to make the statements look nice, we introduce

Definition 4.1. Let A be a self-adjoint operator, $E_0 \in \mathbb{R}$, $\varepsilon > 0$, and $\eta \in (0, \varepsilon)$. E_0 is an η-approximate ε-isolated eigenvalue of A, if there exists λ such that

$$\sigma(A) \cap [E_0 - \varepsilon, E_0 + \varepsilon] = \{\lambda\}, \tag{4.1}$$

$|E_0 - \lambda| \leqslant \eta$, and λ is simple.

A convenient choice for us will be $\eta = \varepsilon^{10}$ for most of this work. However, leaving η as an independent parameter has a big advantage. If we consider

$$\eta < \tilde{\eta} < \tilde{\varepsilon} < \varepsilon,$$

then we have that η-approximate ε-isolated implies $\tilde{\eta}$-approximate $\tilde{\varepsilon}$-isolated. Similarly to Definition 3.3, we will define what it means for an approximately isolated eigenvalue to extend an eigenvalue.

Definition 4.2. We say that an isolated eigenvalue E_0 of $H^{[-M,M]}$ η-approximately extends to an ε-isolated eigenvalue of $H^{[-R,R]}$ if:

(i) There exists an eigenvalue λ of $H^{[-R,R]}$ satisfying $|\lambda - E_0| \leqslant \eta$.
(ii) $\sigma(H^{[-R,R]}) \cap [E_0 - \varepsilon, E_0 + \varepsilon] = \{\lambda\}$.
(iii) Let ψ be the eigenfunction of $H^{[-M,M]}$ and φ be the one of $H^{[-R,R]}$. Then for some $|a| = 1$

$$\|\psi - a\varphi\|_W \leqslant \eta. \tag{4.2}$$

We note that this definition implies that

$$E_0 \text{ is an } \eta\text{-approximate } \varepsilon\text{-isolated eigenvalue of } H^{[-R,R]}. \tag{4.3}$$

However, as noted in the last section, condition (iii) is crucial to control various quantities needed in our multi-scale scheme. At the end of this section, we will prove

Theorem 4.3. *Given $M_1 \geqslant 1$, there exists $h_3 = h_3(f, M_1, \delta) > 0$ such that for $0 < h < h_3$, $E_0 \in \mathcal{E}_\delta$, we have the following: There exists $\mathfrak{X}_1 \subseteq \mathbb{T}$ satisfying*

$$|\mathfrak{X}_1| \geqslant \frac{1000}{\sqrt{M_1}} \tag{4.4}$$

such that for $x \in \mathfrak{X}_1$, we have with $\varepsilon_1 = h^{\frac{1}{500}}$ and $\eta_1 = h^{\frac{1}{4}}$ that for $x \in \mathfrak{X}_1$

$$E_0 \ \eta_1\text{-approximately extends to an } \varepsilon_1\text{-isolated eigenvalue of } H^{[-M_1,M_1]}_{h,x,\xi_0(x)}. \tag{4.5}$$

We recall from (3.3) that $\xi_0(x) = y_0$ and we understand E_0 as an eigenvalue of $H^{[-0,0]}_{h,x,\xi_0}$. In the following, we will refer to the fact described in the previous theorem as $(\xi_0, \mathfrak{X}_0)$ extends to an η_1-approximate ε_1-parametrization on scale M_1 on $\mathfrak{X}_1$. It should be clear how to generalize this definition to the more general situation, we are interested in. Furthermore, one can check that Lemma 3.4 remains valid in this setting. We now come to the last result, we need to prove the initial condition. I already formulate it in the way, we will need it for the inductive step.

Theorem 4.4. *Let $(\xi, \mathfrak{X})$ be an (ε, L)-parametrization of E_0 for $H^{[-M,M]}_{h,\bullet}$ such that*

$$(\xi, \mathfrak{X}) \text{ is a } \frac{d^5}{20\|f'\|_{L^\infty(\mathbb{T})}}\text{-extension of } (\xi_0, \mathfrak{X}_0). \tag{4.6}$$

Assume

$$(\xi, \mathfrak{X}) \text{ extends to an } \varepsilon^5 \cdot \eta\text{-approximate } \varepsilon\text{-parametrization on scale } R \text{ on } \widetilde{\mathfrak{X}} \tag{4.7}$$

and $\hat{L} = L + \varepsilon \leqslant \frac{1}{3}$. Then there exists $(\hat{\xi}, \widehat{\mathfrak{X}})$ such that

$$(\hat{\xi}, \widehat{\mathfrak{X}}) \text{ is a } \left(\frac{\varepsilon}{2}, \hat{L}\right)\text{-parametrization of } E_0 \text{ for } H^{[-R,R]}_{h,\bullet}, \tag{4.8}$$

$|\widehat{\mathfrak{X}}| \geqslant \frac{1}{3}|\widetilde{\mathfrak{X}}|$, and

$$(\hat{\xi}, \widehat{\mathfrak{X}}) \text{ is a } 2\eta\text{-extension of } (\xi, \mathfrak{X}) \text{ from scale } M \text{ to scale } R. \tag{4.9}$$

The proof of this theorem will be given in Sections 9 and 10. Before giving the proof of Theorem 4.3, we will give the proof of Theorem 3.6.

Proof of Theorem 3.6. This follows from the previous two theorems. □

We now begin the proof of Theorem 4.3. The eigenfunctions of $H^{[-M,M]}_{h,x,y}$ are approximately given by

$$\psi^0_{h,x,y}(n) = \begin{cases} 1, & n = 0; \\ \frac{h}{E_0 - V_{x,y}(-1)}, & n = -1; \\ \frac{h}{E_0 - V_{x,y}(1)}, & n = 1; \\ 0, & \text{otherwise}. \end{cases} \tag{4.10}$$

Define $\psi_{h,x,y} = \frac{1}{\|\psi^0_{h,x,y}\|}\psi^0_{h,x,y}$, the normalized version of the above vector. It should be noted that in order for (4.10) to make sense, we need that $V_{x,y}(\pm 1) \neq 0$. We will be able to ensure this with the following lemma

Lemma 4.5. *Let $M \geqslant 1$, $y_0 \in \mathbb{T}$, and $E_0 = f(y_0)$. There exists $h_4 = h_4(M, f) > 0$ such that for $0 < h < h_4$, we have*:

(i) *There exists* $\mathcal{X}$ *of measure* $|\mathcal{X}| \geqslant \frac{1}{2}$.
(ii) *For* $x \in \mathcal{X}$, $n \in [-M, M] \setminus \{0\}$, *we have*

$$\left|V_{x,y_0}(n) - E_0\right| \geqslant h^{\frac{1}{1000}}. \tag{4.11}$$

In order to prove this lemma, we need to recall some things about analytic functions. Since f is analytic there exist $F > 0$ and $\alpha > 0$ such that for every $E \in \mathbb{R}$ and $\varepsilon > 0$

$$\left|\left\{x \in \mathbb{T}\colon \left|f(x) - E\right| < \varepsilon\right\}\right| \leqslant F \cdot \varepsilon^{\alpha}. \tag{4.12}$$

Since $V_{x,y}(n) = f(y + nx + n(n-1)\alpha)$, this implies that

$$\left|\left\{x \in \mathbb{T}\colon \left|V_{x,y}(n) - E\right| < \varepsilon\right\}\right| \leqslant F \cdot \varepsilon^{\alpha} \tag{4.13}$$

for all $E \in \mathbb{R}$, $\varepsilon > 0$, $y \in \mathbb{T}$, $n \in \mathbb{Z} \setminus \{0\}$. For $n = 0$, (4.13) fails, since $V_{x,y}(0) = f(y)$.

Proof of Lemma 4.5. By (4.13), we can find a set $\mathcal{X}$ such that for $x \in \mathcal{X}$, we have

$$\left|V_{x,y_0}(n) - E_0\right| \geqslant h^{\frac{1}{1000}}$$

for $n \in [-N, N] \setminus \{0\}$ and

$$|\mathcal{X}| \geqslant 1 - 2NF \cdot (h)^{\frac{\alpha}{1000}},$$

so $|\mathcal{X}| \geqslant \frac{1}{2}$ for $h \leqslant (\frac{1}{4NF})^{\frac{1000}{\alpha}}$. $\square$

Lemma 4.5 implies

Lemma 4.6. *Let* $x \in \mathcal{X}$, *then*

$$\left\|\left(H_{h,x,y_0}^{[-M,M]} - E_0\right)\psi_{h,x,y}^0\right\| = \sqrt{6}h^{1-\frac{1}{1000}}, \qquad 1 \leqslant \left\|\psi_{h,x,y}^0\right\| \leqslant 1 + h^{2-\frac{1}{500}}. \tag{4.14}$$

Proof. We have $|\psi_{h,x,y}^0(\pm 1)| \leqslant h^{1-\frac{1}{1000}}$. The first inequality follows by the some computations, for the second one notice $\|\psi_{h,x,y}^0\| \leqslant \sqrt{1 + 2h^{2(1-\frac{1}{1000})}}$, which implies the claim since $\sqrt{1+t} \leqslant 1 + \frac{t}{2}$. $\square$

Hence, we have that $H_{h,x,y_0}^{[-M,M]}$ has an eigenvalue λ_x, that satisfies $|\lambda_x - E_0| \leqslant \sqrt{h}$, possibly imposing a new smallness condition on h. Denote by E_j the eigenvalues of $H_{h,x,y_0}^{[-M,M]}$ and by φ_j the corresponding eigenfunctions. The previous lemma implies that there exists ℓ such that

$$|E_0 - E_\ell| \leqslant \sqrt{h}. \tag{4.15}$$

Lemma 4.7. *Assume* h *is small enough. Then for* $j \neq \ell$

$$|E_0 - E_j| \geqslant h^{\frac{1}{500}}. \tag{4.16}$$

Proof. Consider the operator $\widehat{H}$ which is defined to be equal to $H_{h,x,y_0}^{[-M,M]}$, except that we replace $V(0)$ by $42 + E_0$. We have that

$$\sigma(\widehat{H}) \cap \left[E_0 - h^{\frac{1}{1000}} + h, E_0 + h^{\frac{1}{1000}} - h\right] = \emptyset.$$

Since $H_{h,x,y_0}^{[-M,M]} - \widehat{H}$ is a rank one operator, the claim follows. $\square$

We summarize the findings so far as

$$E_0 \text{ is a } \sqrt{h}\text{-approximate } h^{\frac{1}{500}}\text{-isolated eigenvalue of } H_{h,x,y_0}^{[-M,M]} \tag{4.17}$$

for $x \in \mathcal{X}$. It remains to check condition (iii) of Definition 4.2. Since the φ_j form an orthonormal basis, we can write $\psi_{h,x,y_0} = \sum_j \langle \varphi_j, \psi_{h,x,y_0}\rangle \varphi_j$. We have

Lemma 4.8. *For $x \in \mathcal{X}$, we have that*

$$\sum_{j \neq \ell} \left|\langle \varphi_j, \psi_{h,x,y_0}\rangle\right|^2 \leqslant h^{\frac{1}{2}}. \tag{4.18}$$

Proof. A computation shows that for $x \in \mathcal{X}$

$$\sum_j |E_j - E_0|^2 \left|\langle \varphi_j, \psi_{h,x,y_0}\rangle\right|^2 = \left\|\left(H_{h,x,y_0}^{[-M,M]} - E_0\right)\psi_{h,x,y_0}\right\|^2 \leqslant 6h^{2(1-\frac{1}{1000})}.$$

By the previous lemma, we have for $j \neq \ell$ that $|E_j - E_0|^2 \geqslant \frac{1}{2}h^{\frac{1}{2}}$. The claim follows by some computations. $\square$

We now come to

Proof of Theorem 4.3. The previous lemma implies that

$$\|\varphi_\ell - a\psi_{h,x,y}\| \leqslant h^{\frac{1}{4}}$$

for some $|a| = 1$. This finishes the proof. $\square$

5. The proof of the multi-scale step; Theorem 3.5

I will begin by introducing the notion of *suitability* for a Schrödinger operator H acting on $\ell^2(\mathbb{Z})$, and then introduce the large deviation estimates. After this, I will discuss the proof of Theorem 3.5. Given an interval $[a,b] \subseteq \mathbb{Z}$, we introduce $H^{[a,b]}$ as the restriction of H to $\ell^2(\{a,\dots,b\})$. For $E \in \mathbb{R}$ and $k, \ell \in [a,b]$, we introduce the *Green's function* by

$$G^{[a,b]}(E,k,\ell) = \left\langle e_k, \left(H^{[a,b]} - E\right)^{-1} e_\ell\right\rangle, \tag{5.1}$$

where $\{e_\ell\}_{\ell\in\mathbb{Z}}$ denotes the standard basis of $\ell^2(\mathbb{Z})$, that is

$$e_\ell(n) = \begin{cases} 1, & \ell = n; \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

We will quantify the properties of the Green's function with the following definition.

Definition 5.1. Let $\gamma > 0$, $\Gamma > 1$, and $p \geqslant 0$. An interval $[-N, N]$ is called (γ, Γ, p)-suitable if the following hold:

(i) $\Gamma \leqslant \gamma N$.
(ii) We have

$$\left\| \left(H^{[-N,N]} - E\right)^{-1} \right\| \leqslant \frac{1}{2^p} \mathrm{e}^{\Gamma}. \tag{5.3}$$

(iii) For $k \in \{-N, N\}$ and $-\frac{2}{3}N \leqslant \ell \leqslant \frac{2}{3}N$, we have

$$\left| G^{[-N,N]}(E, k, \ell) \right| \leqslant \frac{1}{2^p} \mathrm{e}^{-\gamma|k-\ell|}. \tag{5.4}$$

Lemma 6.1 shows that this definition has a certain stability under perturbing H. The next definition is again for the specific operator $H_{h,x,y}$ defined in (2.6).

Definition 5.2. Let $\gamma > 0$, $\Gamma > 1$, $p \geqslant 0$, $N \geqslant 1$, and $E \in \mathbb{R}$. The set of unsuitability $\mathcal{U}^{[-N,N]}_{h,E,\gamma,\Gamma,p}$ denotes the set of all $(x, y) \in \mathbb{T}^2$ such that

$$[-N, N] \text{ is not } (\gamma, \Gamma, p)\text{-suitable for } H_{h,x,y} - E. \tag{5.5}$$

Lemma 6.2 will derive a certain geometric structure for the set $\mathcal{U}$, whereas the next theorem shows that the measure of this set is small.

Theorem 5.3. *Assume that the Diophantine condition* (2.1) *holds. There exist* $h_0 = h_0(f, c) > 0$, $\gamma \geqslant 1$ *such that for* $N \geqslant 100$, $E \in \mathbb{R}$, *and* $0 < h < h_0$

$$\left| \mathcal{U}^{[-N,N]}_{h,E,\gamma,\frac{1}{2}\gamma N,5} \right| \leqslant \mathrm{e}^{-N^{\frac{1}{10}}}. \tag{5.6}$$

Proof. A proof of this theorem can be extracted from [8,7,10]. □

We will refer to the assumption that the conclusions of the previous theorem hold, as that *the large deviation estimates* hold. The specific form of the constants in the theorem above is of little importance to us, in particular the exponent $\frac{1}{10}$ in $\mathrm{e}^{-N^{\frac{1}{10}}}$ could be any number > 0. I am using this concrete value to reduce the number of constants in the proofs. The next theorem is the basic conclusion, we will draw from it.

Theorem 5.4. *Assume that the large deviation estimates hold, and that* M *is large enough. Let* $N = \lfloor M^{\frac{1}{100}} \rfloor$, $R = \lfloor \mathrm{e}^{M^{\frac{1}{1000}}} \rfloor$, *and* $\xi : \mathbb{T} \to \mathbb{T}$ *satisfying* $\|\xi'\|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{3}$. *Then there exists* $\mathcal{B} \subseteq \mathbb{T}$ *satisfying*

$$|\mathcal{B}| \leqslant \frac{1}{M^{\frac{3}{4}}} \tag{5.7}$$

such that for $\frac{M}{10} \leqslant |n| \leqslant R$, *we have*

$$n + [-N, N] \text{ is } (\gamma, \gamma N, 2)\text{-suitable for } H_{h,x,\xi(x)} - E_0 \tag{5.8}$$

for $x \in \mathbb{T} \setminus \mathcal{B}$.

It should be pointed out here, that this theorem does not tell us that double resonances happen with small probability. It tells us that resonances happen with small probability along curves satisfying certain estimates. We will use these curves to parametrize resonances of $H_{h,x,y}^{[-M,M]}$. This way, we can eliminate the double resonances relevant to us. It would be interesting to obtain a true double resonance elimination theorem, since it would imply Anderson localization. I will still refer to the previous result as *double resonance elimination*.

I also wish to point out that the geometric content of Theorem 5.4 should be surprising. The information that the large deviation estimates hold, tells us that the measure of a subset of $\mathbb{T}^2$ is small, then the output tells us that certain curves intersect this set with small probability. This is possible since, the set $\mathcal{U}$ has further geometric structure and the availability of a fast variable.

Having eliminated double resonances, we have the following result, which tells us that eigenvalues extend

Theorem 5.5. *Let* $R \geqslant M \geqslant N$, $40\mathrm{e}^{-\frac{1}{5}\gamma M} \leqslant \varepsilon \leqslant \mathrm{e}^{-3\gamma N}$. *Assume for* $\frac{M}{10} \leqslant |n| \leqslant R$, *we have*

$$n + [-N, N] \text{ is } (\gamma, \gamma N, 2)\text{-suitable for } H - E_0 \tag{5.9}$$

and

$$E_0 \text{ is an } \varepsilon\text{-isolated eigenvalue of } H^{[-M,M]}. \tag{5.10}$$

Then for $\eta = 2\mathrm{e}^{-\frac{\gamma}{5}M}$

$$E_0 \text{ extends to an } \eta\text{-approximate } \frac{\varepsilon}{1000}\text{-isolated eigenvalue of } H^{[-R,R]}. \tag{5.11}$$

The proof of this theorem directly follows from the more abstract Theorem 8.1. We have now provided the abstract methods for proving Theorem 3.5. We need

Lemma 5.6. *Let* ξ *be as in Theorem* 3.5. *We have that*

$$\|\xi'\|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{3}. \tag{5.12}$$

Proof. We have that $\xi_0'(x) = 0$ for all x. Since $(\xi, \mathfrak{X})$ is a $\frac{1}{3}$-extension of $(\xi_0, \mathfrak{X}_0)$, the claim follows. $\square$

Proof of Theorem 3.5. Apply Theorem 5.4 and introduce

$$\mathfrak{X}_1 = \mathfrak{X} \setminus \mathcal{B}, \qquad |\mathfrak{X}_1| \geqslant \frac{1}{2}|\mathfrak{X}|.$$

By Theorem 5.5, we have for $x \in \mathfrak{X}_1$ and $\eta = \mathrm{e}^{-\frac{1}{5}M}$ that

$$(\xi, \mathfrak{X}) \text{ extends to an } \eta\text{-approximate } \frac{\varepsilon}{1000}\text{-parametrization of } E_0 \text{ for } H_{h,\bullet}^{[-R,R]}.$$

By Theorem 4.4, we obtain a parametrization for $(\hat{\xi}, \widehat{\mathfrak{X}})$ such that

$$|\widehat{\mathfrak{X}}| \geqslant \frac{1}{6}|\mathfrak{X}|.$$

The claims follow using that $\|\varphi\|_W \leqslant (1 + R^2)\|\varphi\|$. $\square$

6. Suitability

In this section, we discuss the notion of suitability defined in Definition 5.1 in more detail. We begin with the following stability result

Lemma 6.1. *Assume* $\|\widetilde{H}^{[-N,N]} - H^{[-N,N]}\| \leqslant \frac{1}{2^{p+2}}\mathrm{e}^{-3\gamma N}$ *and*

$$[-N, N] \text{ is } (\gamma, \Gamma, p+1)\text{-suitable for } H - E. \tag{6.1}$$

Then

$$[-N, N] \text{ is } (\gamma, \Gamma, p)\text{-suitable for } \widetilde{H} - E. \tag{6.2}$$

Proof. This is Lemma 5.3 in [20]. $\square$

For the mechanism to eliminate double resonances, we will need to understand the horizontal slices of the set of unsuitability. For $y \in \mathbb{T}$, we introduce

$$U(y) = \{x\colon\ (x, y) \in U\} \tag{6.3}$$

for any set $U \subseteq \mathbb{T}^2$. We will need

Lemma 6.2. *For N large enough, there exists a set U such that*

$$\mathcal{U}^{[-N,N]}_{h,E,\gamma,\frac{1}{2}\gamma N,3} \subseteq U \subseteq \mathcal{U}^{[-N,N]}_{h,E,\gamma,\gamma N,5} \tag{6.4}$$

and for $y \in \mathbb{T}$

$$U(y) \text{ consists of less then } N^{10} \text{ intervals.} \tag{6.5}$$

Denote by $\hat{f}(k)$ the Fourier coefficients of f, that is

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e(k \cdot x)$$

with $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$. Introduce $f^R(x) = \sum_{k=-R}^{R} \hat{f}(k) e(k \cdot x)$ and by $H_{h,x,y,R}$ the operator with potential

$$V_{x,y,R}(n) = f^R\big(\big(T^n(x,y)\big)_2\big) = f^R\big(y + nx + n(n-1)\alpha\big). \tag{6.6}$$

Since f is analytic, we have for some positive $c > 0$ that $\|f - f^R\|_{L^\infty(\mathbb{T})} \leqslant \mathrm{e}^{-cR}$ for R large enough. Define U to be the set of all x such that:

(i) We have

$$\big\|\big(H_{h,x,y,R}^{[-N,N]} - E\big)^{-1}\big\|_{\mathrm{HS}} > \frac{1}{2^p} \mathrm{e}^{\frac{1}{2}\gamma N}. \tag{6.7}$$

(ii) Condition (iii) of Definition 5.1 holds.

Here $\|.\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm, that is

$$\|A\|_{\mathrm{HS}} = \left(\sum_{k=1}^{L} \sum_{\ell=1}^{L} |A_{k,\ell}|^2\right)^{\frac{1}{2}}, \tag{6.8}$$

where A is an $L \times L$ matrix. We note that

$$\|A\| \leqslant \|A\|_{\mathrm{HS}} \leqslant L\|A\|. \tag{6.9}$$

Choosing $R = N^2$, the inclusions of the sets follow by Lemma 6.1.

Proof of Lemma 6.2. It remains to discuss the bound on the number of intervals of the sections $U(y)$. By construction and choice of R, we have that for $|n| \leqslant N$

$$\deg\big(V_{x,y,R}(n)\big) \leqslant N^3,$$

where deg(.) denotes the degree of a trigonometric polynomial in x. Since, using Cramer's rule, one can rewrite the conditions defining U as less than $10N$ conditions involving polynomials of degree $\leqslant 5N$ in the $V_{x,y,R}(n)$, the claim follows. □

7. Elimination of the fast variable

In this section, I discuss a variant of the frequency elimination argument from the work [9] by Bourgain and Goldstein. There are two differences. First for us y will not be fixed, but depend on x. Second, we have some additional terms, since we vary x, the fast variable of the skew-shift, and not the frequency of a rotation.

Proposition 7.1. *Let $U \subseteq \mathbb{T}^2$ and assume that for $y \in \mathbb{T}$, we have that*

$$U(y) = \{x\colon (x, y) \in U\} \tag{7.1}$$

consists of at most M intervals. Furthermore let $\xi : \mathbb{T} \to \mathbb{T}$ be a continuously differentiable function satisfying for $x \in \mathbb{T}$

$$\left|\xi'(x)\right| \leqslant \frac{1}{3}. \tag{7.2}$$

Then for $R \geqslant 1$, we have

$$\left|\left\{x \in \mathbb{T}\colon \exists\, \ell \sim R\colon T^\ell\big(x, \xi(x)\big) \in U\right\}\right| \leqslant 120R^4\sqrt{|U|} + \frac{2M}{R}. \tag{7.3}$$

Here we denote by $\ell \sim R$ that $R \leqslant \ell \leqslant 2R$. If one were to relax (7.2) to $|\xi'(x)| \leqslant L$, one would need to replace the meaning of this to $\ell = R + 3r\lceil L\rceil$ for $0 \leqslant r \leqslant R - 1$. For simplicity, I have decided to work with (7.2).

Before proving Proposition 7.1, we show how it implies Theorem 5.4.

Proof of Theorem 5.4. Apply the previous proposition to the set U constructed in Lemma 6.2. We have that

$$|U| \leqslant \mathrm{e}^{-M^{\frac{1}{1000}}}$$

and that the sections $U(y)$ consist of less then $M^{\frac{1}{10}}$ many intervals. Define a sequence R_j by

$$R_j = 2^{j-1}\frac{M}{10}$$

for $j = 1, \dots, j_{\max}$, where $j_{\max}$ is defined to be minimal such that $R_{j_{\max}} > R$. We can apply Proposition 7.1 to U and these R_j and see that the claim holds as long as M is large enough. □

The main idea of the proof of Proposition 7.1 is that the second coordinate of $T^\ell(x, \xi(x))$ is ℓ to 1, whereas the first one is 1 to 1. This create sufficient independence of the two coordinates to imply the claim.

We begin now with fleshing out the details. Define for $x \in \mathbb{T}$ and $\ell \sim R$

$$\big(\varphi_\ell(x), \psi_\ell(x)\big) = T^\ell\big(x, \xi(x)\big), \tag{7.4}$$

so that

$$\varphi_\ell(x) = x + 2\ell\alpha, \qquad \psi_\ell(x) = \xi(x) + \ell x + \ell(\ell - 1)\alpha.$$

We will need

Lemma 7.2. *The map ψ_ℓ is ℓ to 1 and satisfies for $x \in \mathbb{T}$*

$$\ell - \frac{1}{3} \leqslant \psi_\ell'(x) \leqslant \ell + \frac{1}{3}. \tag{7.5}$$

Proof. Since $|\xi'(x)| \leqslant \frac{1}{3}$, we have that for any $x, y \in \mathbb{T}$

$$\bigl|\xi(x) - \xi(y)\bigr| \leqslant \frac{1}{3}.$$

The claim about ψ_ℓ being ℓ to 1 follows. The claim about the derivative is a computation. □

By the previous lemma, there exist maps $\theta_{\ell,p} : \mathbb{T} \to \mathbb{T}$ such that for every $x \in \mathbb{T}$, there exists a unique $1 \leqslant p \leqslant \ell$ such that

$$x = \theta_{\ell,p}\bigl(\psi_\ell(x)\bigr). \tag{7.6}$$

From this, we have for these that

$$\frac{1}{\ell + \frac{1}{3}} \leqslant \theta_{\ell,p}'(y) \leqslant \frac{1}{\ell - \frac{1}{3}} \tag{7.7}$$

and that for any $y \in \mathbb{T}$ and $1 \leqslant p \leqslant \ell$

$$\psi_\ell\bigl(\theta_{\ell,p}(y)\bigr) = y. \tag{7.8}$$

Lemma 7.3. *We have*

$$\begin{aligned} &\bigl|\bigl\{x \in \mathbb{T}\colon \exists\, \ell \sim R\colon T^\ell\bigl(x, \xi(x)\bigr) \in U\bigr\}\bigr| \\ &\quad \leqslant \frac{2}{R} \int_{\mathbb{T}} \#\left\{ \begin{matrix} R \leqslant \ell \leqslant 2R \\ 1 \leqslant p \leqslant \ell \end{matrix} \colon \varphi_\ell\bigl(\theta_{p,\ell}(y)\bigr) \in U(y) \right\} dy. \end{aligned} \tag{7.9}$$

Proof. Since

$$\sum_{\ell=R}^{2R} \chi_U\bigl(\varphi_\ell(x), \psi_\ell(x)\bigr) = \#\bigl\{\ell \sim R\colon T^\ell\bigl(x, \xi(x)\bigr) \in U\bigr\}$$

we have

$$\bigl|\bigl\{x \in \mathbb{T}\colon \exists \ell \sim R\colon T^\ell\bigl(x, \xi(x)\bigr) \in U\bigr\}\bigr| \leqslant \sum_{\ell=R}^{2R} \int_{\mathbb{T}} \chi_U\bigl(\varphi_\ell(x), \psi_\ell(x)\bigr)\, dx.$$

Performing the change of variables $y = \psi_\ell(x)$, we obtain that

$$\leqslant \frac{2}{R} \sum_{\ell=R}^{2R} \sum_{p=1}^{\ell} \int_{\mathbb{T}} \chi_U\big(\varphi_\ell\big(\theta_{\ell,p}(y)\big), y\big)\, dy.$$

The claim follows. $\square$

A simple computation shows the estimate

$$\#\left\{\begin{matrix} R \leqslant \ell \leqslant 2R \\ 1 \leqslant p \leqslant \ell \end{matrix}\colon\ \varphi_\ell\big(\theta_{p,\ell}(y)\big) \in U(y)\right\} \leqslant 2R^2. \tag{7.10}$$

Given $\gamma > 0$, define the set $\mathcal{B}_1^\gamma$ by

$$\mathcal{B}_1^\gamma = \big\{y\colon\ \big|U(y)\big| > \gamma\big\}. \tag{7.11}$$

By Markov's inequality, we have that $|\mathcal{B}_1^\gamma| \leqslant \frac{1}{\gamma}|U|$ and

$$\int_{\mathcal{B}_1^\gamma} \#\left\{\begin{matrix} R \leqslant \ell \leqslant 2R \\ 1 \leqslant p \leqslant \ell \end{matrix}\colon\ \varphi_\ell\big(\theta_{p,\ell}(y)\big) \in U(y)\right\} dy \leqslant \frac{2}{\gamma}|U| \cdot R^2. \tag{7.12}$$

Define $\eta_{\ell,p}(y) = \varphi_\ell(\theta_{\ell,p}(y))$. Define the set $\mathcal{B}_2^\gamma$ as the set of y such that there exist $(\ell_1, p_1) \neq (\ell_2, p_2)$ such that

$$\big|\eta_{\ell_1,p_1}(y) - \eta_{\ell_2,p_2}(y)\big| \leqslant \gamma. \tag{7.13}$$

Lemma 7.4. *For $\gamma < \frac{1}{100R}$, we have that*

$$\big|\mathcal{B}_2^\gamma\big| \leqslant 200R^6\gamma. \tag{7.14}$$

Proof. Consider for $(\ell_1, p_1) \neq (\ell_2, p_2)$ the set $\mathcal{Y}_{(\ell_1,p_1),(\ell_2,p_2)}$ of y satisfying (7.13). If $\ell_1 = \ell_2$, we have that $\mathcal{Y}_{(\ell_1,p_1),(\ell_2,p_2)} = \emptyset$, because of the constructions of the functions $\theta_{\ell,p}$.

So consider now $\ell_1 < \ell_2$ and define

$$g(y) = \eta_{\ell_1,p_1}(y) - \eta_{\ell_2,p_2}(y).$$

A computation using (7.7) shows that $|g'(y)| \geqslant \frac{1}{20R^2}$. This implies that

$$|\mathcal{Y}_{(\ell_1,p_1),(\ell_2,p_2)}| \leqslant 40R^2\gamma$$

since the functions $\theta_{p,\ell}$ are increasing.

Now, since there are less than $4R^4$ possible choices for $(\ell_1, p_1) \neq (\ell_2, p_2)$ the claim follows. $\square$

The previous considerations imply that if we choose

$$\gamma = \frac{1}{10R^3}\sqrt{|U|}, \tag{7.15}$$

we obtain that

$$\int\limits_{\mathcal{B}_1^\gamma \cup \mathcal{B}_2^\gamma} \#\left\{\begin{matrix} R \leqslant \ell \leqslant 2R \\ 1 \leqslant p \leqslant \ell \end{matrix} \colon \varphi_\ell\big(\theta_{p,\ell}(y)\big) \in U(y)\right\} dy \leqslant 60R^5\sqrt{|U|}. \tag{7.16}$$

The last piece is

Lemma 7.5. *Let $y \in \mathbb{T} \setminus (\mathcal{B}_1^\gamma \cup \mathcal{B}_2^\gamma)$, then*

$$\#\left\{\begin{matrix} R \leqslant \ell \leqslant 2R \\ 1 \leqslant p \leqslant \ell \end{matrix} \colon \varphi_\ell\big(\theta_{p,\ell}(y)\big) \in U(y)\right\} \leqslant M. \tag{7.17}$$

Proof. By assumption and $y \notin \mathcal{B}_1^\gamma$, we may write

$$U(y) = \bigcup_{j=1}^{\widehat{M}} I_j$$

for $\widehat{M} \leqslant M$ and I_j intervals of length $\leqslant \gamma$. Since $y \notin \mathcal{B}_2^\gamma$, we have that for each j there is at most one pair (ℓ, p) such that

$$\varphi_\ell\big(\theta_{p,\ell}(y)\big) \in I_j.$$

The claim follows. □

Proof of Proposition 7.1. This follows by the previous lemma and the equation preceding it. □

8. Controlling a single eigenvalue

This section develops a mechanism to keep control of a single eigenvalue, when changing from scale to scale. The basic idea is the following: A single of the restrictions of the operator has an eigenvalue in a given energy interval, here $H^{[-M,M]}$ in the energy interval $[E_0 - \varepsilon, E_0 + \varepsilon]$. If all other restrictions are suitable for this energy interval, then the same holds for the larger interval.

Theorem 8.1. *Let $R \geqslant M \geqslant N \geqslant 1$, $E_0 \in \mathbb{R}$, and $\varepsilon > 0$. Assume:*

(i) *For $n \in [-R, R] \setminus [-\frac{M}{10}, \frac{M}{10}]$ with $[n - N, n + N] \subseteq [-R, R]$, we have*

$$[n - N, n + N] \text{ is } (\gamma, \gamma N, 1)\text{-suitable for } H - E_0. \tag{8.1}$$

(ii) $40\mathrm{e}^{-\frac{1}{5}\gamma M} \leqslant \varepsilon \leqslant \mathrm{e}^{-3\gamma N}$.
(iii) $H^{[-M,M]}$ *has exactly one eigenvalue* $\lambda_0 \in [E_0 - \varepsilon, E_0 + \varepsilon]$.
(iv) *This eigenvalue satisfies*

$$|\lambda_0 - E_0| \leqslant \frac{\varepsilon}{100}. \tag{8.2}$$

Then $H^{[-R,R]}$ *has exactly one eigenvalue* E *in* $[E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$, *which satisfies*

$$|E - \lambda_0| \leqslant 2\mathrm{e}^{-\frac{\gamma}{5}M}, \qquad |E - E_0| \leqslant \frac{\varepsilon}{10}. \tag{8.3}$$

Furthermore, let φ *be a normalized eigenfunction of* $H^{[-M,M]}$ *corresponding the eigenvalue* λ_0. *Then there exists a normalized eigenfunction* ψ *of* $H^{[-R,R]}$ *corresponding to the eigenvalue* E *such that*

$$\|\varphi - \psi\| \leqslant 2\mathrm{e}^{-\frac{1}{10}\gamma M}. \tag{8.4}$$

Before starting with the proof of this theorem, let us note

Corollary 8.2. *Let* φ *and* ψ *be as in the previous theorem. Then for any bounded operator* A, *we have*

$$\bigl|\langle\psi, A\psi\rangle - \langle\varphi, A\varphi\rangle\bigr| \leqslant 4\|A\|\mathrm{e}^{-\frac{1}{10}\gamma M}. \tag{8.5}$$

Proof. Write $\eta = \psi - \varphi$. We have $\|\eta\| \leqslant 2\mathrm{e}^{-\frac{1}{10}\gamma M}$. A computation shows that

$$\langle\psi, A\psi\rangle - \langle\varphi, A\varphi\rangle = \langle\eta, A\psi\rangle + \langle\varphi, A\eta\rangle.$$

Since φ and ψ are normalized, the claim follows. □

It should be pointed out at this point that the quantity $\langle\psi, A\psi\rangle$ is independent of the normalized eigenfunction chosen. This follows from that for two normalized eigenfunctions $\psi, \tilde{\psi}$ of $H^{[-R,R]}$ to the eigenvalue E, one has $\psi = c\tilde{\psi}$ for some $|c| = 1$, since the eigenspaces of $H^{[-R,R]}$ to a single eigenvalue are one-dimensional.

Let us now begin with actually proving Theorem 8.1. The following lemma is a consequence of Lemma 6.1.

Lemma 8.3. *Let* $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$ *and* $n \in [-R, R] \setminus [-\frac{M}{10}, \frac{M}{10}]$ *with* $[n - N, n + N] \subseteq [-R, R]$. *Then*

$$[n - N, n + N] \text{ is } (\gamma, \gamma N, 0)\text{-suitable for } H - E. \tag{8.6}$$

Having this lemma, we will now first show that $H^{[-R,R]}$ has at least one eigenvalue in the interval $[E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$. This follows by the next lemma, since $2\mathrm{e}^{-\frac{1}{5}\gamma M} \leqslant \frac{\varepsilon}{20}$.

Lemma 8.4. *Let $h \in (0, \frac{1}{2})$. Then there exists an eigenvalue E of $H^{[-R,R]}$ in*

$$\left[\lambda_0 - 2\mathrm{e}^{-\frac{1}{5}\gamma M}, \lambda_0 + 2\mathrm{e}^{-\frac{1}{5}\gamma M}\right]. \tag{8.7}$$

Proof. We recall that $H^{[-M,M]}\varphi = \lambda_0 \varphi$. For $n \in [-(N-M), M-N] \setminus [-\frac{1}{10}M, \frac{1}{10}M]$, we have by Lemma 8.3

$$\left|\varphi(n)\right| \leqslant \mathrm{e}^{-\gamma N} \max\left(\left|\varphi(n+N+1)\right|, \left|\varphi(n-N-1)\right|\right),$$

where $\varphi(n) = 0$ for $n \notin [-M, M]$. By iterating this equation, we find since $\frac{9}{10}M \cdot \frac{N}{N+1} \leqslant \frac{1}{4}$ that

$$\left|\varphi(M)\right|, \left|\varphi(-M)\right| \leqslant \frac{1}{4}\mathrm{e}^{-\frac{1}{5}\gamma M}.$$

By using φ as a test function for $H^{[-R,R]}$, we have

$$\left\|\left(H^{[-R,R]} - \lambda\right)\varphi\right\| \leqslant \mathrm{e}^{-\frac{1}{5}\gamma M}.$$

This implies the claim. □

We now turn to show that $H^{[-R,R]}$ has only one eigenvalue in the energy interval $[E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$. Let now $E \in [E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$ be an eigenvalue of $H^{[-R,R]}$ and ψ a corresponding normalized eigenfunction. Define the function u by

$$u(n) = \begin{cases} \psi(n), & -M \leqslant n \leqslant M; \\ 0, & \text{otherwise}. \end{cases} \tag{8.8}$$

We have

Lemma 8.5. *Let $h \in (0, \frac{1}{2})$. Then*

$$\left\|\left(H^{[-M,M]} - E\right)u\right\| \leqslant \mathrm{e}^{-\frac{\gamma}{5}M} \tag{8.9}$$

and $\|u\|_{\ell^2([-M,M])} \geqslant 1 - 10\mathrm{e}^{-\frac{\gamma}{5}M}$.

Proof. By Lemma 8.3, we have for $n \in [-R-M, R-M] \setminus [-\frac{M}{10}, \frac{M}{10}]$ that

$$\left|\psi(n)\right| \leqslant \mathrm{e}^{-\gamma N} \max\left(\left|\psi(n+N+1)\right|, \left|\psi(n-N-1)\right|\right)$$

where $\psi(n) = 0$ for $|n| \geqslant R+1$. By iterating this, we can conclude that ψ decays exponentially away from $[-\frac{M}{10}, \frac{M}{10}]$. In fact, one obtains that

$$\left|\psi(n)\right| \leqslant \mathrm{e}^{-\tilde{\gamma}(|n| - \frac{1}{10}M)},$$

where $\tilde{\gamma} = \gamma(1 - \frac{1}{N}) \geqslant \frac{\gamma}{2}$. (8.9) follows as in the previous lemma. The claim on the norm of u is easy. □

We may choose eigenvalue λ_j and eigenfunctions φ_j of $H^{[-M,M]}$ for $j \in [-M, M] \setminus \{0\}$, which complete λ_0 and $\varphi_0 = \varphi$ to an orthonormal basis of $\ell^2(\{-M, \dots, M\})$. Define $u_1 = \frac{1}{\|u\|} u$, which is now a normalized function in $\ell^2(\{-M, \dots, M\})$, which satisfies

$$\big\| \big(H^{[-M,M]} - E \big) u_1 \big\| \leqslant 20 \mathrm{e}^{-\frac{\gamma}{5} M} \tag{8.10}$$

by the previous lemma. We have that

$$\psi = \sum_{j=-M}^{M} \langle \varphi_j, u_1 \rangle \varphi_j. \tag{8.11}$$

We have that

Lemma 8.6. *For* $j \neq 0$, *we have*

$$\big| \langle \varphi_j, u_1 \rangle \big| \leqslant 20 \mathrm{e}^{-\frac{\gamma}{10} M}. \tag{8.12}$$

Proof. We have that

$$\sum_j (E_j - E)^2 \big| \langle \varphi_j, \psi \rangle \big|^2 = \big\| \big(H^{[-M,M]} - E \big) u_1 \big\| \leqslant 400 \mathrm{e}^{-2\frac{\gamma}{5} M}.$$

Now the claim follows, since for $j \neq u_1$, we have $|E_j - E| \geqslant \frac{1}{2} \varepsilon \geqslant \mathrm{e}^{-\frac{\gamma}{10} N}$. □

Hence, we have that

$$\big| \langle \varphi_0, u_1 \rangle \big|^2 = 1 - \sum_{j \neq 0} \big| \langle \varphi_j, u_1 \rangle \big|^2 \geqslant 1 - 40 M \mathrm{e}^{-\frac{1}{10} \gamma M}. \tag{8.13}$$

Proof of Theorem 8.1. Since

$$\langle \varphi_0, \psi \rangle = \|u\| \langle \varphi_0, u_1 \rangle,$$

we may achieve by replacing ψ by $c\psi$, where $c = \frac{\overline{\langle \varphi_0, u_1 \rangle}}{|\langle \varphi_0, u_1 \rangle|}$ that

$$\big| \langle \varphi_0, \psi \rangle - 1 \big| \leqslant \mathrm{e}^{-\frac{1}{20} \gamma M}.$$

Since

$$\|\psi - \varphi_0\|^2 = \|\psi\|^2 + \|\varphi_0\|^2 - 2 \langle \varphi_0, \psi \rangle$$

the claim of the theorem follows. □

9. Eigenvector perturbations

In this section, I discuss how to parametrize eigenvalues in a small neighborhood. I have decided to work in a somewhat general setting, in hope that this clarifies some of the arguments. We will be considering a continuously differentiable family of self-adjoint operator $H(x,y)$ defined for $(x,y) \in [-t,t]^2$. We will always assume that

$$\left\| \partial_x H(x,y) \right\| \leqslant C, \qquad \left\| \partial_y H(x,y) \right\| \leqslant C \tag{9.1}$$

for $(x,y) \in [-t,t]^2$. We will be interested in small perturbations, that is in the language above that t is small. We begin with

Theorem 9.1. *Let $\varepsilon > \eta > 0$ with $4\eta < \varepsilon$. Assume* (9.1) *and that*

$$E_0 \text{ is an } \eta\text{-approximate } \varepsilon\text{-isolated eigenvalue of } H(0,0). \tag{9.2}$$

Define $s = \min(t, \frac{1}{64}\frac{\eta\varepsilon}{C})$. Then for any $(x,y) \in [-s,s]^2$, we have

$$E_0 \text{ is a } 2\eta\text{-approximate } \frac{1}{2}\varepsilon\text{-isolated eigenvalue of } H(x,y). \tag{9.3}$$

Furthermore denote by $\psi_{x,y}$ the associated normalized eigenfunction of $H(x,y)$. Then for any $(x,y), (\tilde{x},\tilde{y}) \in D_s$, there exists $|a| = 1$ such that

$$\|\psi_{x,y} - a\psi_{\tilde{x},\tilde{y}}\| \leqslant \eta. \tag{9.4}$$

We recall that E_0 is an η-approximate ε-isolated eigenvalue of $H(0,0)$, if there exists an eigenvalue λ of $H(0,0)$ such that

$$\sigma\big(H(0,0)\big) \cap [E_0 - \varepsilon, E_0 + \varepsilon] = \{\lambda\} \tag{9.5}$$

and $|\lambda - E_0| \leqslant \eta$. See Definition 4.1.

Instead of proving Theorem 9.1 directly, I will first prove the following more abstract lemma.

Lemma 9.2. *Let A, B be self-adjoint operators, $E_0 \in \mathbb{R}$, $\varepsilon > 0$. Assume that E is a simple eigenvalue of A such that*

$$\sigma(A) \cap [E_0 - \varepsilon, E_0 + \varepsilon] = \{E\}, \qquad |E - E_0| \leqslant \frac{1}{4}\varepsilon \tag{9.6}$$

and that $\|A - B\| \leqslant t\varepsilon$ with $t \in (0, \frac{1}{4})$. Then there exists a simple eigenvalue λ of B such that $|\lambda - E| \leqslant t\varepsilon$ and

$$\sigma(B) \cap \left[E_0 - \frac{3}{4}\varepsilon, E_0 + \frac{3}{4}\varepsilon\right] = \{\lambda\}. \tag{9.7}$$

Furthermore, denote by ψ, φ normalized vectors such that $A\psi = E\psi$, $B\varphi = \lambda\varphi$. Then, there *exists* $|a| = 1$ *such that*

$$\|\varphi - a\psi\| \leqslant 15t. \tag{9.8}$$

Proof. Since $\|A - B\| \leqslant t\varepsilon$ and $t \in (0, \frac{1}{4})$, it follows that

$$\{\lambda\} = \sigma(H) \cap \left[E - \frac{3}{4}\varepsilon, E + \frac{3}{4}\varepsilon\right], \qquad |\lambda - E| \leqslant \varepsilon t.$$

Define $\lambda_1 = \lambda$ and denote by $\lambda_2, \dots, \lambda_M$ an enumeration of the other eigenvalues. Denote by $\varphi_1, \dots, \varphi_M$ a choice of corresponding normalized eigenfunctions. We have

$$\left\|(B - \lambda)\psi\right\| \leqslant \left\|(B - A)\psi\right\| + \left\|(\lambda - E)\psi\right\| + \left\|(A - E)\psi\right\| \leqslant \varepsilon t + \varepsilon t + 0 \leqslant 2\varepsilon t$$

and thus

$$\left\|(B - \lambda)\psi\right\|^2 = \sum_{j=2}^{M} (\lambda - \lambda_j)^2 \left|\langle\psi, \varphi_j\rangle\right|^2 \leqslant 4\varepsilon^2 t^2.$$

Furthermore, we have for $j \geqslant 2$

$$|\lambda_j - E_0| \geqslant \frac{3}{4}\varepsilon, \qquad |\lambda - E_0| \leqslant \frac{1}{2}\varepsilon$$

and thus $|\lambda - \lambda_j| \geqslant \frac{1}{4}\varepsilon$. It follows that $\sum_{j=2}^{M} |\langle\psi, \varphi_j\rangle|^2 \leqslant 64t^2$. This implies that $|\langle\psi, \varphi_1\rangle| \geqslant \sqrt{1 - 64t^2} \geqslant 1 - 128t^2$. Choose

$$\varphi = \frac{\overline{\langle\psi, \varphi_1\rangle}}{|\langle\psi, \varphi_1\rangle|} \varphi_1.$$

Then, we have $\|\psi - \varphi\|^2 = 2(1 - \langle\varphi, \psi\rangle) \leqslant 64t^2$. $\square$

We now come to

Proof of Theorem 9.1. A computation shows that for $(x, y) \in D_s$ that

$$\left\|H(x, y) - H(0, 0)\right\| \leqslant C|x| + C|y| \leqslant \frac{1}{32}\varepsilon\eta.$$

The first claim now follows from the previous lemma. The furthermore statement follows from the furthermore statement of the previous lemma and that $\|H(x, y) - H(\tilde{x}, \tilde{y})\| \leqslant \frac{\eta}{16}\varepsilon$. $\square$

Next, we come to

Theorem 9.3. *Let $\varepsilon > \eta > 0$, $\delta \in (0, \frac{1}{3})$. Assume* (9.1) *and* (9.2)*. Furthermore let ψ be a normalized eigenfunction of $H(0,0)$ corresponding to the eigenvalue $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$ and assume*

$$\langle \psi, \partial_y H(0,0)\psi \rangle \geqslant 2\delta, \qquad \left|\langle \psi, \partial_x H(0,0)\psi \rangle\right| \leqslant \frac{1}{2}\delta^2. \tag{9.9}$$

Define

$$s = \min\left(t, \frac{1}{50} \cdot \frac{\varepsilon\delta^2}{C^2}\right) \tag{9.10}$$

and also assume

$$|E - E_0| \leqslant \frac{\delta s}{3}. \tag{9.11}$$

Then for $(x, y) \in D_s$, there exists a curve $\xi : [-s, s] \to \mathbb{R}$ such that

$$E_0 \text{ is a } \frac{\varepsilon}{2}\text{-isolated eigenvalue of } H\big(x, \xi(x)\big), \tag{9.12}$$

$|\xi'(x)| \leqslant \delta$, and

$$\left|\xi(x)\right| \leqslant \frac{|E - E_0|}{\delta} + \delta s. \tag{9.13}$$

We have

Lemma 9.4. *Let $\psi_{x,y}$ be the eigenfunction of $H(x, y)$ associated to the eigenvalue in $[E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$. Then*

$$\langle \psi_{x,y}, \partial_y H(x,y)\psi_{x,y} \rangle \geqslant \delta, \qquad \left|\langle \psi_{x,y}, \partial_x H(x,y)\psi_{x,y} \rangle\right| \leqslant \delta^2. \tag{9.14}$$

Proof. Since $|x|, |y| \leqslant \frac{1}{24}\frac{\varepsilon}{C} \cdot \frac{\delta^2}{2C}$, we have by Theorem 9.1 for some $|a| = 1$ that

$$\|\psi_{x,y} - a\psi_{0,0}\| \leqslant \frac{\delta^2}{2C}.$$

Furthermore a computation shows for any normalized φ, ψ and operator A

$$\left|\langle \psi, A\psi \rangle - \langle \varphi, A\varphi \rangle\right| \leqslant 2\|A\| \cdot \|\psi - \varphi\|.$$

Now, the claim follows by (9.9). $\square$

It is well known that the unique eigenvalue $\lambda(x, y)$ of $H(x, y)$ in $[E_0 - \frac{\varepsilon}{2}, E_0 + \frac{\varepsilon}{2}]$ is a continuously differentiable function of x and y. Furthermore, its derivatives are given by

$$\partial_x \lambda(x,y) = \langle \psi_{x,y}, \partial_x H(x,y)\psi_{x,y} \rangle, \qquad \partial_y \lambda(x,y) = \langle \psi_{x,y}, \partial_y H(x,y)\psi_{x,y} \rangle. \tag{9.15}$$

Lemma 9.5. *For $|x| \leqslant s$, there exists $\xi(x)$ such that*

$$\lambda\big(x, \xi(x)\big) = E_0. \tag{9.16}$$

Proof. By (9.11) and the previous lemma, we have that

$$\big|\lambda(x, 0) - E_0\big| \leqslant \frac{2}{3}\delta s.$$

Again by the previous lemma, we thus obtain that

$$\lambda(x, s) \geqslant E_0 + \frac{1}{3}\delta s, \qquad \lambda(x, -s) \geqslant E_0 - \frac{1}{3}\delta s.$$

The claim follows since $\lambda(x, y)$ is continuous. □

We are now ready for

Proof of Theorem 9.3. The intermediate value theorem implies that there exists some $\xi(x)$ such that $\lambda(x, \xi(x)) = E_0$. It follows from the implicit function theorem that

$$\xi'(x) = -\frac{\langle \psi_{x,y}, \partial_x H(x, y)\psi_{x,y}\rangle}{\langle \psi_{x,y}, \partial_y H(x, y)\psi_{x,y}\rangle}.$$

The claim follows by some more computations. □

Theorem 9.3 is not good enough for our purposes, since we will need a better estimate on $\|\xi\|_{L^\infty([-s,s])}$. We will prove

Theorem 9.6. *Let $\varepsilon > \eta > 0$, $\delta \in (0, \frac{1}{3})$. Assume* (9.1), (9.2)*, and* (9.9)*, and define s by* (9.10)*. Furthermore let $\mathfrak{X} \subseteq [-s, s]$ and assume that for $x \in \mathfrak{X}$, we have*

$$\big|\lambda(x, 0) - E_0\big| \leqslant \delta\eta. \tag{9.17}$$

Then, we can choose an absolutely continuous function ξ such that $|\xi'(x)| \leqslant \delta$, $|\xi(x)| \leqslant \eta$, and for $x \in \mathfrak{X}$, we have

$$E_0 \textit{ is a } \frac{\varepsilon}{2}\textit{-isolated eigenvalue of } H\big(x, \xi(x)\big). \tag{9.18}$$

Let ξ_0 be given by the previous theorem. For $x \in \mathfrak{X}$, we can compute

$$\big|\lambda\big(x, \xi(x)\big) - \lambda(x, 0)\big| = \big|\lambda(x, 0) - E_0\big| \leqslant \delta\eta$$

and

$$\left|\lambda\big(x,\xi(x)\big)-\lambda(x,0)\right| \geqslant \left|\int_0^{\xi(x)} \lambda_y(x,\tau)\,d\tau\right| \geqslant \delta\left|\xi(x)\right|,$$

which implies $|\xi(x)| \leqslant \eta$.

Proof of Theorem 9.6. Define $a = \inf(\mathfrak{X})$, $b = \sup(\mathfrak{X})$ and introduce

$$\xi(x) = \begin{cases} \xi_0(a), & x \leqslant a; \\ \xi_0(x), & x \in [a,b]; \\ \xi_0(b), & x \geqslant b. \end{cases}$$

The claims now follow by some computations. □

10. Proof of Theorem 4.4

In order to prove Theorem 4.4, we will need the following proposition, which allows us to construct continuously differentiable functions $\mathbb{T} \to \mathbb{T}$.

Proposition 10.1. *Given $L_0 > 0$, $\varepsilon > 0$, $\delta > 0$, $\mathfrak{X} \subseteq \mathbb{T}$, and a continuously differentiable function $\hat{\xi}_0 : \mathbb{T} \to \mathbb{T}$ satisfying for $x \in \mathbb{T}$*

$$\left|\hat{\xi}_0'(x)\right| \leqslant L_0. \tag{10.1}$$

Let furthermore $I_1, \dots, I_Q \subseteq \mathbb{T}$ be disjoint intervals satisfying $|I_q| \geqslant \varepsilon$ and $\xi_q : I_q \to \mathbb{T}$ be continuously differentiable functions satisfying for $x \in I_q$

$$\left|\xi_q'(x)\right| \leqslant L_0, \qquad \left|\xi_q(x) - \hat{\xi}_0(x)\right| < \delta. \tag{10.2}$$

Then there exists a subset $\mathcal{Q} \subseteq \{1, \dots, Q\}$ such that

$$\left|\bigcup_{q\in\mathcal{Q}} I_q \cap \mathfrak{X}\right| \geqslant \frac{1}{3}\left|\bigcup_{q=1}^{Q} I_q \cap \mathfrak{X}\right| \tag{10.3}$$

and a continuously differentiable function $\xi : \mathbb{T} \to \mathbb{T}$ satisfying for $q \in \mathcal{Q}$ that for $x \in I_q$ we have $\xi(x) = \xi_q(x)$ and the bound

$$\left\|\xi'\right\|_{L^\infty(\mathbb{T})} \leqslant L_0 + 3\frac{\delta}{\varepsilon}, \qquad \|\xi - \hat{\xi}_0\|_{L^\infty(\mathbb{T})} \leqslant 5\delta. \tag{10.4}$$

We begin by constructing the set $\mathcal{Q}$. For intervals $I, \widetilde{I} \subseteq \mathbb{T}$, we write

$$\operatorname{dist}(I, \widetilde{I}) = \inf_{x\in I,\ \tilde{x}\in\widetilde{I}} \|x - \tilde{x}\|. \tag{10.5}$$

Lemma 10.2. *There exists $\mathcal{Q}$ such that*:

(i) *For $q, \tilde{q} \in \mathcal{Q}$ with $q \neq \tilde{q}$, we have*

$$\operatorname{dist}(I_q, I_{\tilde{q}}) \geqslant \varepsilon. \tag{10.6}$$

(ii) *We have*

$$\left|\bigcup_{q\in\mathcal{Q}} I_q\right| \geqslant \frac{1}{3}\left|\bigcup_{q=1}^{Q} I_q\right|. \tag{10.7}$$

Proof. Write $I_q = [a_q, b_q]$ and order the intervals such that

$$0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_Q < b_Q.$$

We may have $b_Q > 0$. For $Q = 2P$ even define the three sets

$$\mathcal{Q}_1 = \{2p,\ 1 \leqslant p \leqslant P\}, \qquad \mathcal{Q}_2 = \{2p-1,\ 1 \leqslant p \leqslant P\}, \qquad \mathcal{Q}_3 = \emptyset.$$

For $Q = 2P + 1$ odd, define

$$\mathcal{Q}_1 = \{2p,\ 1 \leqslant p \leqslant P\}, \qquad \mathcal{Q}_2 = \{2p-1,\ 1 \leqslant p \leqslant P\}, \qquad \mathcal{Q}_3 = \{Q\}.$$

Since $|I_q| \geqslant \varepsilon$ and we ordered the intervals, property (i) holds. Furthermore, since the sets $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are disjoint, there exists $j \in \{1,2,3\}$ such that property (ii) holds. Chose $\mathcal{Q} = \mathcal{Q}_j$, finishing the proof. □

Define $J = \bigcup_{q\in\mathcal{Q}} I_q$ and introduce for $x \in J$

$$\eta_1(x) = \xi_q(x) - \hat{\xi}_0(x), \quad \text{if } x \in I_q. \tag{10.8}$$

By assumption, we now have that $\|\eta_1\|_{L^\infty(J)} \leqslant \eta$ and $\|\eta_1'\|_{L^\infty(J)} \leqslant L_0$. We now have that

Lemma 10.3. *There exists a continuously differentiable map $\eta : \mathbb{T} \to \mathbb{T}$ such that $\eta(x) = \eta_1(x)$ for $x \in J$ and $\|\eta'\|_{L^\infty(\mathbb{T})} \leqslant L + 3\frac{\delta}{\varepsilon}$.*

Proof. We can write $J = \bigcup_{p=1}^{P} [a_p, b_p]$ with

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_P < b_P.$$

We then have that $|a_{p+1} - b_p| \geqslant \varepsilon$ and that $|\eta_1(b_p) - \eta_1(a_{p+1})| \leqslant \delta$. The claim then follows by some computations. □

Proof of Proposition 10.1. Define $\xi = \hat{\xi}_0 + \eta$. The claim follows by some more computations. □

Having established this preliminary proposition, we will now proceed to prove Theorem 4.4. The first step is to apply Theorem 9.3. Define $Q = \lceil \frac{50R^2 \cdot \|f'\|^2_{L^\infty(\mathbb{T})}}{\varepsilon d^2} \rceil$ and a sequence of intervals

$$I_q = \left[\frac{q-1}{Q}, \frac{q}{Q}\right) \tag{10.9}$$

for $q = 1, \dots, Q$. We have that the I_q partition $\mathbb{T}$. Call q *good*, if $I_q \cap \widetilde{\mathfrak{X}} \neq \emptyset$.

Lemma 10.4. *We have that*

$$\left| \bigcup_{q \text{ good}} I_q \cap \widetilde{\mathfrak{X}} \right| = |\widetilde{\mathfrak{X}}|. \tag{10.10}$$

Proof. It is easy to see that $\widetilde{\mathfrak{X}} \subseteq \bigcup_{q \text{ good}} I_q$, which implies the claim. $\square$

For each good q, we choose $x_q \in I_q \cap \widetilde{\mathfrak{X}}$ and define

$$H_q(x, y) = H^{[-R,R]}_{h, x_q + x, \xi(x_q + x) + y}. \tag{10.11}$$

We note that, we have that

$$\left\| \partial_x H_q(x, y) \right\| \leqslant R \cdot \left\| f' \right\|_{L^\infty(\mathbb{T})}, \qquad \left\| \partial_y H_q(x, y) \right\| \leqslant \left\| f' \right\|_{L^\infty(\mathbb{T})}. \tag{10.12}$$

We have

Lemma 10.5. *Let d be defined as in* (3.9). *Then for ψ the eigenfunction of $H(0, 0)$ corresponding to the eigenvalue in the interval $[E_0 - \varepsilon, E_0 + \varepsilon]$, we have*

$$\left| \langle \psi, \partial_x H_q \psi \rangle \right| \leqslant \frac{1}{2} d^5, \qquad \left| \langle \psi, \partial_y H_q \psi \rangle \right| \geqslant 2d. \tag{10.13}$$

Proof. By Lemma 3.4, we have that

$$\left| \langle \psi_1, V_x \psi_1 \rangle \right| \leqslant \frac{1}{4} d^5, \qquad \left| \langle \psi_1, V_y \psi_1 \rangle \right| \geqslant 4d,$$

with ψ_1 the eigenfunction of $H^{[-M,M]}_{h, x_q, \xi(x_q)}$. The claim now follows by the extension property from $H^{[-M,M]}$ to $H^{[-R,R]}$. $\square$

We thus have that the conditions of Theorem 9.1 and Theorem 9.3 hold. Using these, we obtain

Lemma 10.6. *Let q be good. Then there exists a curve $\xi_q : I_q \to \mathbb{T}$ such that for $x \in I_q \cap \widetilde{\mathfrak{X}}$, we have*

$$E_0 \ 2\eta\text{-extends to a } \frac{\varepsilon}{2}\text{-isolated eigenvalue of } H^{[-R,R]}_{h, x, \xi_q(x)}. \tag{10.14}$$

Furthermore, ξ_q satisfies:

(i) *For $x \in I_q$, we have $|\xi_q'(x)| \leqslant \frac{1}{10}$.*
(ii) *For $x \in I_q \cap \mathfrak{X}$, we have*

$$\left|\xi_q(x) - \xi(x)\right| \leqslant \eta. \tag{10.15}$$

(iii) *For $x \in \partial(I_q)$, we have $|\xi_q(x) - \xi(x)| \leqslant \eta$.*

Proof. By Theorem 9.6, we obtain for each good q a map $\tilde{\xi}_q$ defined on the interval $[-\frac{1}{Q}, \frac{1}{Q}]$ such that

$$E_0 \text{ is a } \frac{\varepsilon}{2}\text{-isolated eigenvalue of } H\big(x, \tilde{\xi}_q(x)\big).$$

Define $\xi_q(x) = \tilde{\xi}_q(x - x_q) + \xi(x)$. It is easy to check that the previous equation implies for $x \in I_q$ that

$$E_0 \text{ is a } \frac{\varepsilon}{2}\text{-isolated eigenvalue of } H^{[-R,R]}_{h,x,\tilde{\xi}_q(x)}.$$

Furthermore, we have that $|\xi_q(x) - \xi(x)| \leqslant \eta$. We will now check (10.14). First, one can describe the conclusions in the furthermore of Theorem 9.1 as

$$E_0 \text{ extends to a } \frac{\varepsilon}{2}\text{-isolated eigenvalue of } H\big(x, \tilde{\xi}_q(x)\big) \text{ from } H(x, y)$$

for any y. This combined with assumption of Theorem 4.4 implies (10.14). □

Proof of Theorem 4.4. Proposition 10.1 allows us to construct a function $\hat{\xi}$ such that for $x \in \mathbb{T}$

$$\left|\hat{\xi}'(x)\right| \leqslant L + \varepsilon, \qquad \left|\hat{\xi}(x) - \xi(x)\right| \leqslant 2\eta$$

and for $q \in \mathcal{Q}$, $x \in \mathfrak{X} \cap I_q$, we have the claimed properties of the eigenvalue E_0 and the eigenfunctions. □

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Table 1
$h = 1$.

N	Density of the spectrum
320	0.291089
640	0.231054
1280	0.174700
2560	0.139430
5120	0.063408
10 240	0.025548
20 480	0.013934
40 960	0.009152
81 920	0.003842
163 840	0.002419

Appendix A. Numerical evidence

The essential problem in computing the spectrum of a Schrödinger operator H acting on $\ell^2(\mathbb{Z})$ is that this is an infinite dimensional space, so we will need to approximate the spectrum of H by $H^{[-N,N]}$ for some large N. The essential insight is that the difference

$$H - H^{(-\infty,-N-1]} \oplus H^{[-N,N]} \oplus H^{[N+1,\infty)} \tag{A.1}$$

is a rank 4 operator. Hence, if an interval $[a,b]$ contains more than 5 eigenvalues of $H^{[-N,N]}$ then

$$\sigma(H) \cap [a,b] \neq \emptyset. \tag{A.2}$$

The conclusion of this is that, if we denote by

$$E_{-N}^{[-N,N]} < E_{-N+1}^{[-N,N]} < \cdots < E_{N-1}^{[-N,N]} < E_{N}^{[-N,N]} \tag{A.3}$$

the eigenvalues of $H^{[-N,N]}$ and define

$$\delta^{[-N,N]} = \inf_{-N \leqslant j \leqslant N-5} \left(E_{j+5}^{[-N,N]} - E_{j}^{[-N,N]}\right), \tag{A.4}$$

then we have that the spectrum of H is at least $\delta^{[-N,N]}$-dense in $[E_{-N+4}^{[-N,N]}, E_{N-4}^{[-N,N]}]$.

In Table 1, I show the results for

$$V(n) = 2\cos\left(2\pi\sqrt{2}n^2\right) \tag{A.5}$$

for $h = 1$. It should be noted that since the density decreases with a similar rate as N grows, one should expect the spectrum of $\Delta + V$ to be an interval.

Appendix B. The top and bottom of the spectrum

In this appendix, I show how to obtain bounds at the top E_+ of the spectrum. It is easy to see that a similar bound is valid for the bottom of the spectrum. We have that

$$E_+(h) = \sup\bigl(\sigma(H_{h,x,y})\bigr) = \sup_{\|\varphi\|=1} \langle \varphi, H_{h,x,y}\varphi\rangle \tag{B.1}$$

where $H_{h,x,y}$ was defined in (2.6) and the second inequality follows from the minimax principle. We have that

$$\begin{aligned}\langle \varphi, H_{h,x,y}\varphi\rangle &= \sum_{n\in\mathbb{Z}}\bigl(h\cdot\varphi(n)\bigl(\varphi(n+1)+\varphi(n-1)\bigr)+V_{x,y}(n)\varphi(n)^2\bigr),\\ V_{x,y}(n) &= f\bigl(y+nx+n(n-1)\alpha\bigr).\end{aligned} \tag{B.2}$$

It is easy to see that $E_+(h) \leqslant \max(f) + 2h$. Choose now y_0 such that $f(y_0) = \max(f)$ and consider

$$\varphi_0(n) = \begin{cases} \frac{1}{\sqrt{2}}, & n \in \{0,1\};\\ 0, & \text{otherwise.}\end{cases} \tag{B.3}$$

A computation shows that

$$\|\varphi_0\| = 1, \qquad \langle \varphi_0, H_{h,0,y_0}\varphi_0\rangle = f(y_0) + h. \tag{B.4}$$

Hence, we have obtained that

$$\max(f) + h \leqslant E_+(h) \leqslant \max(f) + 2h, \tag{B.5}$$

which is all we claimed.

References

[1] A. Avila, J. Bochi, D. Damanik, Cantor spectrum for Schrödinger operators with potentials arising from generalized skew-shifts, Duke Math. J. 146 (2009) 253–280.

[2] A. Avila, J. Bochi, D. Damanik, Opening gaps in the spectrum of strictly ergodic Schrödinger operators, J. Eur. Math. Soc. (JEMS), in press.

[3] A. Avila, S. Jitomirskaya, Solving the ten martini problem, Lecture Notes in Phys. 690 (2006) 5–16.

[4] A. Avila, S. Jitomirskaya, The ten martini problem, Ann. of Math. 170 (2009) 303–342.

[5] J. Bourgain, On the spectrum of lattice Schrödinger operators with deterministic potential, J. Anal. Math. 87 (2002) 37–75.

[6] J. Bourgain, On the spectrum of lattice Schrödinger operators with deterministic potential (II), J. Anal. Math. 88 (2002) 221–254.

[7] J. Bourgain, Estimates on Green's functions, localization and the quantum kicked rotor model, Ann. of Math. (2) 156 (1) (2002) 249–294.

[8] J. Bourgain, Green's Function Estimates for Lattice Schrödinger Operators and Applications, Ann. of Math. Stud., vol. 158, Princeton University Press, Princeton, NJ, 2005, x+173 pp.

[9] J. Bourgain, M. Goldstein, On nonperturbative localization with quasi-periodic potential, Ann. Math. 152 (2000) 835–879.

[10] J. Bourgain, M. Goldstein, W. Schlag, Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift, Comm. Math. Phys. 220 (3) (2001) 583–621.
[11] V. Chulaevsky, Y. Sinai, Anderson localization for the I-D discrete Schrödinger operator with two-frequency potential, Comm. Math. Phys. 125 (1989) 91–112.
[12] D. Damanik, D. Lenz, Zero-measure Cantor spectrum for Schrödinger operators with low-complexity potentials, J. Math. Pures Appl. 85 (2006) 671–686.
[13] D. Damanik, D. Lenz, A criterion of Boshernitzan and uniform convergence in the multiplicative ergodic theorem, Duke Math. J. 133 (2006) 95–123.
[14] Z. Gan, H. Krüger, Discontinuity of the Lyapunov exponent, preprint.
[15] M. Goldstein, W. Schlag, On the formation of gaps in the spectrum of Schrödinger operators with quasi-periodic potentials, in: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, in: Proc. Sympos. Pure Math., vol. 76, Part 2, Amer. Math. Soc., Providence, RI, 2007, pp. 539–563.
[16] M. Goldstein, W. Schlag, On resonances and the formation of gaps in the spectrum of quasi-periodic Schrödinger equations, Ann. of Math. (2) 173 (1) (2011) 337–475.
[17] W. Kirsch, An invitation to random Schrödinger operators, with an appendix by Frédéric Klopp, in: Random Schrödinger Operators, in: Panor. Synthèses, vol. 25, Soc. Math. France, Paris, 2008, pp. 1–119.
[18] H. Krüger, A family of Schrödinger operators whose spectrum is an interval, Comm. Math. Phys. 290 (3) (2009) 935–939.
[19] H. Krüger, Semiclassical analysis of the largest gap of quasi-periodic Schrödinger operators, Math. Model. Nat. Phenom. 5 (4) (2010) 256–268.
[20] H. Krüger, Localization for random operators with non-monotone potentials with exponentially decaying correlations, preprint.
[21] R. Nichols, G. Stolz, Spectral properties of the discrete random displacement model, preprint.
[22] J. Pöschel, Examples of discrete Schrödinger operators with pure point spectrum, Comm. Math. Phys. 88 (1983) 447–463.
[23] Z. Rudnick, P. Sarnak, A. Zaharescu, The distribution of spacings between the fractional parts of $n^2\alpha$, Invent. Math. 145 (2001) 37–57.
[24] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Math. Surveys Monogr., vol. 72, Amer. Math. Soc., RI, 2000.